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Currents and Super-Operator Product Expansions of $N = 4$ Super-W Algebra

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The currents and the super-operators product expansions of the $N = 4$ super-W algebra are obtained. The generators of this extended conformal algebra consist of the stress-tensor superfield J and the primary superfields $\Phi^q_{\pm\nu}$ of integer or half-integer conformal weight Δ and Cartan–Weyl charge q. The algorithm for deriving the $N = 4$ super-W algebra is given for particular cases. Explicit forms of the operator product algebra $\Phi\Phi$ for different values of Δ are given.

1. INTRODUCTION

There are two main reasons for studying extended symmetries in conformal field theory. The first is that certain applications of conformal field theory (in string theory or statistical mechanics) [1–3] require some extra symmetry in addition to conformal invariance. The second reason is that extended symmetries can be used to facilitate the analysis of a large class of conformal field theories (called rational conformal field theories) [1, 4] and eventually to classify certain types of conformal field theories. Different approaches have been employed in the study of extended conformal symmetries such as the direct construction developed in refs. 5–8, the Drinfeld–Sokolov approach [9–12], and the dual formalism [13].

The algebraic structure that emerges in the study of extended symmetry is a higher spin extension of the Virasoro algebra, which is commonly called a W-algebra. As defined by Zamolodchikov [5], this algebra contains a conserved current $\phi(z)$ of integer or half-integer spin Δ in addition to a stress tensor T. It gives a consistent theory when the associativity of the product is satisfied. The W-algebra can be determined completely by demanding the

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associativity of the operator product with respect to both $T\phi\phi$ and $\phi\phi\phi$. Alternatively, one can construct the W-algebra by demanding first associativity with respect to $T\phi\phi$ [5], which leads to recursion relations [14]. Then one has to check associativity with respect to $\phi\phi\phi$.

In practice, two methods of checking the associativity are known. One method is to examine the crossing symmetry of the four-point function. The other is to use normal-ordered graded commutators defined by Bouwknegt [15]. In this method the associativity of the operator product expansion (OPE) algebra implies the Jacobi identity of the commutator algebra. These methods have been used in the construction of bosonic (first method) [5], $N = 1$, and $N = 2$ supersymmetric W algebra [16, 17, 1].

The present work is based on the direct construction, which consists in writing down an extended algebra by proposing a number of extra generators and closing the algebra. The purpose of the present paper is to determine the currents and the super-operator product expansions (SOPE) of the $N = 4$ supersymmetric extension of Zamolodchikov's W-algebra. The generators of the $N = 4$ super$W$ algebra consist of the stress-tensor superfield J and the primary superfields $\Phi^q_{\pm\nu}$ ($\nu = 0, \ldots, \mu$ for μ integer, $\nu = 1/2, \ldots, \mu$ for μ half-integer, and $\mu = 0, 1/2, \ldots, \Delta$) of integer or half-integer conformal weight Δ and Cartan–Weyl charge q. In general the primary superfields are superconformal tensors $\Phi^q_{i_1 \ldots i_r, j_1 \ldots j_s}$ ($i_1, \ldots, i_1; j_1, \ldots, j_s = 1, 2$) [18] characterized by a conformal weight Δ and a Cartan–Weyl charge q. The superconformal transformations of these tensors are generated by a current superfield $J(z)$ through the relation

$$\delta_E \, \Phi^q_{i_1 \ldots i_r, j_1 \ldots j_s}(Z_2) = \oint_{c_{z_2}} dZ_1 \, E(Z_1) J(Z_1) \, \Phi^q_{i_1 \ldots i_r, j_1 \ldots j_s}(Z_2) \tag{1.1}$$

where $E(Z_1)$ is a parameter superfield and $J(Z_1)\Phi^q_{i_1 \ldots i_r, j_1 \ldots j_s}(Z_2)$ is the SOPE. The superfields $\Phi^q_{\pm\nu}$ are certain combinations of superconformal tensors $\Phi^q_{i_1 \ldots i_r, j_1 \ldots j_s}$.

The paper is organized as follows. In Section 2 we give a summary of $N = 4$ superconformal invariance in superspace. In Section 3 we give the SOPE $J(Z_1)J(Z_2)$ and consequently the (anti) commutator algebra of generators. We will see that the $N = 4$ superconformal algebra is parametrized by two central charges c and c' [8] or suppressing one of the central terms by c and the value of a deformation parameter α. The algebra is reduced to the so-called small $N = 4$ algebra [6, 19], admitting one central extension for $\alpha = \pm 1/2$. For $\alpha = 1/2$ the algebra is generated by the superfield $J_k = D^+ S_k D^- J$ ($k = 1, 2, 3$), where

$$D^+ S_k D^- = (S_k)^{ab} D^+_a S_k D^-_b, \qquad (S_k)^{ab} = (S_k)_{ab}$$

with

$$(S_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (S_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (S_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2}$$

and for $\alpha = -1/2$ it is generated by $J^0 = [D^+, D^-]J$ and $J^{\pm\pm} = D^\pm D^\pm J$. Finally, for $\alpha \neq \pm 1/2$ the algebra is generated by $J_a^\pm = D_a^\pm J$ ($a = 1, 2$). In Section 4 we determine the SOPE $J\Phi^q_{\pm\nu}$ and we find that there are two types of $N = 4$ super-W algebra. We show that α must be equal to $\alpha^\pm(\Delta, \mu, q) \neq (1/2, -1/2)$ for $\mu \neq \Delta$ and to $1/2$ for $\mu = \Delta$. We will see that for the first type ($\mu \neq \Delta$) the superfield $\Phi^q_{\pm\nu}$ is null for $q \neq 0$ and for the second type ($\mu = \Delta$) the component fields W^q, F^q, $H^{q\pm 2}$, and B^q of the superfield $\Phi^q_{\pm\nu}$ are null fields. In Section 5 For the first type ($\mu \neq \Delta$) we give the algorithm for constructing the $N = 4$ super-W algebra in the case of $\mu = 0$ and $q = 0$. The descendant fields are determined up to level $n = 2$ for the superconformal family of the identity operator and up to level $n = 1$ for the superconformal family of the superfield Φ. Notice that for Δ half-integer we have simply the superconformal family of the identity operator. So we have an explicit form of the (SOPE) $\Phi\Phi$ for $\Delta = 1/2$. For the second type ($\mu = \Delta$), we give the (SOPE) $\Phi_i\ \Phi_j$ ($i, j = 1, 2, 3$) for $\Delta = 1$ and $q = 0$. The last section is devoted to the conclusion. In the Appendix we present relations useful for deriving the results in the text.

2. SUPERCONFORMAL INVARIANCE IN $N = 4$ $SU(2)$ SUPERSPACE

Let us start with the $N = 4$ $SU(2)$ extended superspace, where we consider only the holomophic sector, since similar arguments are applicable to the antiholomorphic sector. The supercoordinates are given by

$$Z = (z, \theta^{+a}, \theta^{-a}, U^{\pm b}), \qquad a, b = 1, 2 \tag{2.1}$$

with $z = x^1 + ix^2$ (x^u, $u = 1, 2$; spacetime coordinates) and $\theta^{\pm a}$ are analytic Grassmann variables such that

$$\theta^{\pm a} = \theta^{ba} U_b^+ \tag{2.2}$$

where the variables θ^{ba} belong to the vectorial representation 4 of $SU(2) \otimes SU(2)$ and U_b^+ are harmonic projectors converting the $SU(2)$ symmetry into a $U(1)$ one. They parametrize the $SU(2)/U(1) \approx S^2$ sphere and satisfy the following constraints:

$$U^{\pm a}U_a^{\mp} = \pm 1, \qquad U^{\pm a}U_a^{\pm} = 0$$
$$\overline{U_a^-} = U^{+a}, \qquad U_a^{\pm} = \varepsilon_{ab}U^{\pm b}, \qquad \varepsilon_{12} = \varepsilon^{21} = 1 \tag{2.3}$$
$$\theta^{ab} = U^{+a}\theta^{-b} - U^{-a}\theta^{+b}$$

The covariant derivatives are given by

$$D_a^+ = -\frac{\partial}{\partial\theta^{-a}} + \theta_a^+\partial_z, \qquad D_a^- = \frac{\partial}{\partial\theta^{+a}} + \theta_a^-\partial_z \tag{2.4}$$

which satisfy the following graded algebra:

$$\{D_a^+, D_b^-\} = 2\varepsilon_{ab}\partial_z, \qquad \{D_a^{\pm}, D_b^{\pm}\} = 0 \tag{2.5}$$

For simplicity, we will be interested in this analysis only in the variation of z, $\theta^{\pm a}$ and keep the harmonic variables constant. We define the analytic function on this superspace by the Taylor expansion

$$f^q(Z_1) = \sum_{n=0}^{\infty}\frac{1}{n!}Z_{12}^n\left[1 + \theta_{12}^{+a}D_{a2}^- - \theta_{12}^{-a}D_{a2}^+ - \frac{1}{2}\theta_{12}^{+a}\theta_{12}^{+b}D_{a2}^-, D_{b2}^-\right]$$
$$- \frac{1}{2}\theta_{12}^{-a}\theta_{12}^{-b}D_{a2}^+D_{b2}^+ + \frac{1}{2}\theta_{12}^{+a}\theta_{12}^{-b}[D_{a2}^-, D_{b2}^+]$$
$$- \frac{1}{2}(\theta_{12}^+\theta_{12}^-)\theta_{12}^{+c}[D_{c2}^+, D_2^{-c}]D_{c2}^- + \frac{1}{2}(\theta_{12}^+\theta_{12}^-)\theta_{12}^{+c}[D_{c2}^-, D_2^{+c}]D_{c2}^+$$
$$+ \frac{1}{32}(\theta_{12}^+)^2(\theta_{12}^-)^2[D_2^{+c}, D_{c2}^-][D_{c2}^+, D_2^{-c}]]\ \partial_{z_2}^n f^q(Z_2) \tag{2.6}$$

and the Cauchy theorem

$$\oint dZ_1 \frac{1}{4}\frac{(\theta_{12}^+)^2(\theta_{12}^-)^2}{Z_{12}} f^q(Z_1) = f^q(Z_2) \tag{2.7a}$$

with

$$\oint dZ = \frac{1}{2\pi i}\oint_c dz \int d^2\,\theta^+ \int d^2\,\theta^- \tag{2.7b}$$

The contour of the Z_1 integral encloses the point Z_2. Here Z_{12}, θ_{12}^+, and θ_{12}^- are invariant distance under global superconformal transformation of the $N = 4$ superspace given by

$$Z_{12} = z_{12} - \theta_1^{+a}\theta_{a2}^- + \theta_1^{-a}\theta_{a2}^+,$$
$$z_{12} = z_1 - z_2, \qquad \theta_{12}^{\pm a} = \theta_1^{\pm a} - \theta_2^{\pm a},\ \theta^{\pm}\theta^{\pm} = \theta^{\pm a}\theta_a^{\pm}, \tag{2.8}$$

$$\theta^{+}\theta^{-} = \theta^{+a}\theta^{-}_{a} = \theta^{-a}\theta^{+}_{a}$$

Now we introduce the supergeneralization of the $N = 0$ derivative ∂_z, differential dz, and intrinsic operator $\partial = dz\partial_z$ as follows:

$$\begin{aligned} D^{+a}D^{-}_{a} + D^{-}_{a}D^{+a} &= 4\partial_z \\ d\xi^{+}_{a}\, d\xi^{-a} = dZ &= dz + \theta^{+a}\, d\theta^{-}_{a} - \theta^{-a}\, d\theta^{+}_{a} \\ S_1 = d\xi^{+a}\, D^{-}_{a}, &\qquad S_2 = d\xi^{-a}\, D^{+}_{a} \end{aligned} \tag{2.9}$$

where $d\xi^{\pm a}$ are the half-differentials; together with the superderivatives, they play a crucial role in the study of $N = 4$ superconformal structure. The objects S_1 and S_2 are superconformal scalar operators which are invariant under general supercoordinate transformations $\tilde{Z} = \tilde{Z}(Z)$ [18],

$$\tilde{S}_i = S_i, \qquad i = 1, 2 \tag{2.10}$$

This gives the conformal transformation of the covariant derivatives and the half-differentials,

$$\begin{aligned} D^{+}_{a} = -(D^{+}_{a}\, \tilde{\theta}^{-c})\tilde{D}^{+}_{c}, &\qquad D^{-}_{a} = (D^{-}_{a}\, \tilde{\theta}^{+c})\tilde{D}^{-}_{c} \\ d\tilde{\xi}^{+a} = (D^{-}_{c}\, \tilde{\theta}^{+a})d\xi^{+c}, &\qquad d\tilde{\xi}^{-a} = -\, (D^{+}_{c}\, \tilde{\theta}^{-a})d\xi^{-c} \end{aligned} \tag{2.11}$$

These are homogeneous transformations preserving the Grassmann analyticity. This leads to

$$\begin{aligned} D^{\pm}_{a}\tilde{\theta}^{\pm c} = 0, &\qquad D^{+}_{a}\tilde{z} = \mp\tilde{\theta}^{\pm}_{c}(D^{\pm}_{a}\tilde{\theta}^{\mp c}) \\ (D^{+}_{a}\tilde{\theta}^{-}_{c})(D^{-}_{b}\tilde{\theta}^{+c}) &= \varepsilon_{ab}(\partial_z\tilde{z} + \tilde{\theta}^{-}_{c}\partial_z\tilde{\theta}^{+c} - \tilde{\theta}^{+}_{c}\partial_z\tilde{\theta}^{-c}) \end{aligned} \tag{2.12}$$

It turns out that the infinitesimal transformations that satisfy the condition (2.11) can be expressed in terms of an unrestricted superfield $E(Z)$,

$$\delta Z = E + \frac{1}{2}\,\theta^{-a}\, D^{+}_{a}\, E - \frac{1}{2}\,\theta^{+a}\, D^{-}_{a}\, E, \qquad \delta\theta^{\pm}_{a} = \frac{1}{2}\, D^{\pm}_{a}E \tag{2.13}$$

where we have

$$D^{+}_{a}D^{+}_{b}E = 0 = D^{-}_{a}D^{-}_{b}E \tag{2.14}$$

On the other hand, $N = 4$ superconformal tensors are primary analytic superfields $\Phi^{q}_{i_1\ldots i_r j_1\ldots j_s}$ $(i_1, \ldots, i_r; j_0, \ldots, j_s = 1, 2)$ characterized by a conformal weight Δ and a Cartan–Weyl charge q. They are defined so that their combination with the half-differentials $d\xi^{\pm i}$ is invariant:

$$\Phi^q_{i_1\ldots i_r j_1\ldots j_s}(Z) d\xi^{+j_1} \ldots d\xi^{+i_r}\, d\xi^{-j_1} \ldots d\xi^{-j_s}$$
$$= \tilde{\Phi}^q_{i_1\ldots i_r j_1\ldots j_s}(\tilde{Z}) d\tilde{\xi}^{+i_1} \ldots d\xi^{+i_r}\, d\tilde{\xi}^{-j_1} \ldots d\tilde{\xi}^{-j_s} \tag{2.15a}$$

with

$$q = s - r, \qquad \Delta = \frac{r+s}{2} \tag{2.15b}$$

By using Eqs. (2.11), we get the transformation law of $\Phi^q_{i_1\ldots i_r j_1\ldots j_s}(Z)$, namely

$$\Phi^q_{i_1\ldots i_r j_1\ldots j_s} = (-1)^s\, \tilde{\Phi}^q_{k_1\ldots k_l l_1\ldots l_s}(\tilde{Z})(D^-_{i_1}\tilde{\theta}^{+k_1})$$
$$\ldots (D^-_{i_r}\tilde{\theta}^{+k_r})(D^+_{j_1}\tilde{\theta}^{-l_1}) \ldots (D^+_{j_s}\tilde{\theta}^{-l_s}) \tag{2.16}$$

The infinitesimal transformation of Eq (2.16) reads

$$\delta_E \Phi^q_{i_1\ldots i_r j_1\ldots j_s}(Z) = \left[E\partial_z + \frac{1}{2}(D^{+a}E)D^-_a - \frac{1}{2}(D^{-a}E)D^+_a + \Delta\partial_z E \right] \Phi^q_{i_1\ldots i_r j_1\ldots j_s}(Z)$$
$$+ \frac{1}{4}\sum_{n=1}^{r} [D^-_{i_n}, D^{+k_n}] E \Phi^q_{i_1 i_{n-1} p_n i_{n+1}\ldots i_r j_1\ldots j_s}(Z)$$
$$- \frac{1}{4}\sum_{p=1}^{s} [D^-_{i_p}, D^{+k_p}] E \Phi^q_{i_1\ldots i_r j_1\ldots j_{p-1} k_p j_{p+1}\ldots j_s}(Z) \tag{2.17}$$

These transformations are generated by a suppercurrent $J(Z)$ through the relation

$$\delta_E\, \Phi^q_{i_1\ldots i_r j_1\ldots j_s}(Z_2) = \oint_{c_{z_2}} dZ_1\, E(Z_1)\, J(Z_1)\, \Phi^q_{i_1\ldots i_r j_1\ldots j_s}(Z_2) \tag{2.18}$$

where $J(Z_1)\Phi^q_{i_1\ldots i_r j_1\ldots j_s}(Z_2)$ is the SOPE given by

$$J(Z_1)\, \Phi^q_{i_1\ldots i_r j_1\ldots j_s}(Z_2)$$
$$= \left[\frac{1}{4}\frac{(\theta^+_{12})^2 (\theta^-_{12})^2}{Z_{12}} \partial_{z_2} - \frac{1}{2}\frac{(\theta^+_{12}\, \theta^-_{12})\theta^{+a}_{12}}{Z_{12}} D^-_{a_2} \right.$$
$$\left. - \frac{1}{2}\frac{(\theta^+_{12}\, \theta^-_{12})\theta^{-a}_{12}}{Z_{12}} D^+_{a_2} + \frac{\Delta}{4}\frac{(\theta^+_{12})^2 (\theta^-_{12})^2}{Z^2_{12}} \right] \Phi^q_{i_1\ldots i_r j_1\ldots j_s}$$
$$- \frac{1}{2}\sum_{n=1}^{r} \frac{\theta^+_{i_n 12}\, \theta^{-k_n}_{12}}{Z_{12}} \Phi^q_{i_1\ldots i_{n-1} k_n i_{n+1}\ldots i_r j_1\ldots j_s}(Z)$$
$$+ \frac{1}{2}\sum_{p=1}^{s} \frac{\theta^-_{j_p 12}\, \theta^{+l_p}_{12}}{Z_{12}} \Phi^q_{i_1\ldots i_r j_1\ldots j_{p-1} l_p j_{p+1}\ldots j_s}(Z) \tag{2.19}$$

The superconformal tensors are a subclass of a more general class called superconformal tensor densities [18], which can be defined as follows:

$$\Phi^{q}_{i_1 \ldots i_r j_1 \ldots j_s} = (-1)^s \, (A)^{\Delta - \mu} \, \tilde{\Phi}^{q}_{k_1 \ldots k_r l_1 \ldots l_s}(\tilde{Z})(D^-_{i_1} \, \tilde{\theta}^{+k_1}) \ldots (D^-_{i_r} \, \tilde{\theta}^{+k_r})(D^+_{j_1} \, \tilde{\theta}^{-l_1}) \ldots (D^+_{j_s} \, \tilde{\theta}^{-l_s}) \tag{2.20a}$$

where

$$A = \frac{1}{2} (D^{+a} \, \tilde{\theta}^-_b)(D^-_a \, \tilde{\theta}^{+b}) = (\partial_z \, \tilde{z} + \tilde{\theta}_c^- \, \partial_z \, \tilde{\theta}^{+c} - \tilde{\theta}_c^+ \, \partial_z \, \tilde{\theta}^{-c})$$

$$q = s - r, \qquad \mu = (r + s)/2, \qquad \mu = 0, 1/2, \ldots\ldots, \Delta \tag{2.20b}$$

For the superconformal tensor densities (2.20a) the inifinitesimal transformation and the SOPE with $J(Z)$ are given by Eqs. (2.17) and (2.19), respectively. We read off from (2.19) that the conformal dimension of $J(Z)$ is zero. Note that $J(Z)$ is not a primary, but a descendant field in the conformal family of the identity.

3. $N = 4$ SUPERCONFORMAL ALGEBRA

The transformation law of $J(Z)$ is given by

$$J(Z) = \tilde{J}(\tilde{Z}) + S(Z, \tilde{Z}) \tag{3.1}$$

where $S(Z,\tilde{Z})$ [8] is the super-Schwarzian derivative. The transformation rule (3.1) requires that $S(Z,\tilde{Z})$ satisfies the following group property:

$$S(Z, \tilde{\tilde{Z}}) = S(Z, \tilde{Z}) + S(\tilde{Z}, \tilde{\tilde{Z}}) \tag{3.2}$$

Under the infinitesimal superconformal transformation (2.13) with parameter superfield $E(Z)$ the supercurrent $J(Z)$ transforms as

$$\delta_E J(Z) = \left[E\partial_z + \frac{1}{2} (D^{+a} E) D^-_a - \frac{1}{2} (D^{-a} \, E) D^+_a \right] J(Z) + UE \tag{3.3}$$

where

$$UE(Z) = S\left(Z, z + E + \frac{1}{2} \theta^{-a} \, D^+_a \, E - \frac{1}{2} \theta^{+a} \, D^-_a \, E, \theta^{\pm a} + \frac{1}{2} D^{\pm a} \, E \right)$$

This transformation law is equivalent to the SOPE for the supercurrent $J(Z)$,

$$J(Z_1)J(Z_2) = \left[\frac{1}{4}\frac{(\theta_{12}^+)^2\,(\theta_{12}^-)^2}{Z_{12}}\,\partial_{z_2} - \frac{1}{2}\frac{(\theta_{12}^+\,\theta_{12}^-)\theta_{12}^{+a}}{Z_{12}}\,D_{a_2}^- - \frac{1}{2}\frac{(\theta_{12}^+\,\theta_{12}^-)\,\theta_{12}^{-a}}{Z_{12}}\,D_{a_2}^+\right]J(Z_2) + C(Z_1, Z_2) \tag{3.4}$$

where C is related to U through

$$\oint dZ_1\, C(Z_1, Z_2) = UE(Z_2) \tag{3.5}$$

The superconformal Ward identities for the two-point correlation function and dimensional arguments imply that $C(Z_1,Z_2)$ has the following form [8]:

$$C(Z_1, Z_2) = \frac{c'}{48}\frac{(\theta_{12}^+)^2\,(\theta_{12}^-)^2}{Z_{12}^2} - \frac{c}{12}\log Z_{12} \tag{3.6}$$

The superconformal properties of $J(Z)$ are pathological due to the fact that the conformal dimension of $J(Z)$ is zero, hence nonlocal, similar to the nonlocal pathologies of the dimension-zero scalar field in ordinary CFT. We can avoid these problems by formulating the theory in terms of the vector superfields $J_a^\pm(Z) = D\mathrm{D}_a^\pm J(Z)$, which transform as

$$\begin{aligned}\delta_E J_a^\pm(Z) = {} & \left[E\partial_z + \frac{1}{2}(D^{+a}E)D_a^- - \frac{1}{2}(D^{-a}E)D_a^+ + \frac{1}{2}\partial_z E\right]J_a^\pm(Z) \\ & \pm \frac{1}{8}[D^{+b}, D_b^-]EJ_a^\pm(Z) + \frac{1}{4}(S_k)_{ab}(D^+S_kD^-)EJ^{\pm b}(Z) \\ & - \frac{(c-c')}{12}D_a^\pm\,\partial_z E\end{aligned} \tag{3.7}$$

The SOPE $J(Z_1)J(Z_2)$ is equivalent to the (anti) commmutator algebra of the component field of the supercurrent $J(Z)$, given by

$$\begin{aligned} t_n &= \oint_{c_0} dZ\, Z^{n+1}\, J(Z), & j_n &= \frac{1}{2}\oint_{c_0} dZ\,(\theta^+)^2\,(\theta^-)^2\, Z^{n+1}\, J(Z) \\ K_{a,r}^\pm &= 2\oint_{c_0} dZ\,\theta_a^\pm\, Z^{r+1/2}\, J(Z), & j_{a,r}^\pm &= 2\oint_{c_0} dZ(\theta^+\theta^-)\theta_a^\pm Z^{r-1/2}\, J(Z) \\ U_{k,n} &= 2\oint_{c_0} dZ\,(\theta^+S_k\theta^-)\, Z^n\, J(Z), & U &= 2\oint_{c_0} dZ(\theta^+\theta^-)Z^n J(Z) \\ U_n^{\pm\pm} &= \pm\sqrt{2}\oint_{c_0} dZ\,(\theta^+)^2\, Z^n\, J(Z) \end{aligned} \tag{3.8}$$

with $k = 1, 2, 3$, $\theta^+ S_k \theta^- = (S_k)^{ab}\, \theta_a^+ \theta_b^-$, and $n \in \mathbf{Z}$ and $r \in \mathbf{Y} = \mathbf{Z} + 1/2$ (Neveu–Schwarz sector) or $\mathbf{Z}$ (Ramond sector). We introduce a real parameter α and we define the following set of generators:

$$
\begin{aligned}
L_n &= t_n + \alpha n\,(n+1) j_n, \\
G^{\pm}_{a,r} &= K^{\pm}_{a,r} \mp 2\alpha\left(r + \frac{1}{2}\right) j^{\pm}_{a,r},\ U_{k,n},\ U_n,\ U^{\pm\pm}_n,\ j^{\pm}_{a,r},\ j_n
\end{aligned}
\tag{3.9}
$$

The (anti) commutator algebra of these generators is given by

$$
\begin{aligned}
[L_n, L_m] &= (n-m) L_{n+m} + \frac{C_\alpha}{12}\,(n^3 - n)\,\delta_{n+m,0} \\
[L_n, G^{\pm}_{a,r}] &= \left(\frac{n}{2} - r\right) G^{\pm}_{a,n+r}, \qquad [L_n, U_{k,m}] = -m U_{k,m}, \\
[L_n, U_m] &= -m U_{n+m} \\
[L_n, U^{\pm\pm}_m] &= -m U^{\pm\pm}_m, \qquad [L_n, j^{\pm}_{a,r}] = -\left(\frac{n}{2} + r\right) j^{\pm}_{a,n+r} \\
[L_n, j_m] &= -(n+m)\, l_{n+m} - \frac{C'_\alpha}{6}\,(n+1)\delta_{n+m,0} \\
\{G^{+}_{a,r}, G^{-}_{a,s}\} &= 2\varepsilon_{ab}\, L_{r+s} + \frac{1}{2}\,(r-s)(1+2\alpha)\varepsilon_{ab}\, U_{r+s} \\
&\quad + \frac{1}{2}\,(r-s)(1-2\alpha)(S_k)_{ab}\, U_{k,r+s} \\
&\quad + \frac{C_\alpha}{12}\left(r^2 - \frac{1}{4}\right)\delta_{n+m,0} \\
\{G^{\pm}_{a,r}, G^{\pm}_{a,s}\} &= \pm\,\frac{\varepsilon_{ab}}{\sqrt{2}}\,(r-s) U^{\pm\pm}_{r+s}, \\
[U_n, G^{\pm}_{a,r}] &= \pm G^{\pm}_{a,n+r} + n(1-2\alpha) j^{\pm}_{a,n+r} \\
[U^{\pm\pm}_n, G^{\mp}_{a,r}] &= -\sqrt{2} G^{\pm}_{a,n+r} \mp n\sqrt{2}(1+2\alpha) j^{\pm}_{a,n+r}, \qquad [G^{\pm}_{a,r}, U^{\pm\pm}_m] = 0 \\
[U_{k,n}, G^{\pm}_{a,r}] &= -(-1)^k\,(S_k)_{ab}(G^{\pm b}_{n+r} \pm n(1-2\alpha) j^{\pm b}_{n+r}) \\
\{G^{\pm}_{a,r}, j^{\pm}_{b,s}\} &= \varepsilon_{ab}\,(r+s)\, j_{r+s} \pm \frac{1}{2}\,\varepsilon_{ab}\, U_{r+s} - \frac{1}{2}\,(S_k)_{ab}\, U_{k,r+s}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{C'_\alpha}{3}\left(r + \frac{1}{2}\right) \varepsilon_{ab}\, \delta_{r+s,0} \\
\{G^{\pm}_{a,r}, j^{\mp}_{b,s}\} &= -\frac{\varepsilon_{ab}}{\sqrt{2}}\, U^{\pm\pm}_{r+s}, \qquad [G^{\pm}_{a,r}, j_m] = j^{\pm}_{a,r+m} \\
[U_{k,n}, U_{i,m}] &= -2(-1)^j \varepsilon^{kij}\, U_{j,n+m} - (-1)^k \delta_{ki} \frac{2n}{3}(c + c')\, \delta_{n+m,0} \\
[U_{k,n}, U_m] &= 0, \qquad [U_{k,n}, U^{\pm\pm}_m] = 0, \qquad [U_{k,n}, j_m] = 0 \\
[U_{k,n}, j^{\pm}_{a,r}] &= -(-1)^k (S_k)_{ab}\, j^{\pm b}_{n+r} \\
[U_n, U_m] &= \frac{2n}{3}(c - c')\delta_{n+m,0}, \qquad [U_n, U^{\pm\pm}_m] = \pm 2U^{\pm\pm}_{n+m}, \\
[U_n, j^{\pm}_{a,r}] &= \pm\, j^{\pm}_{a,n+r} \\
[U_n, j_m] &= 0, \qquad [U^{++}_n, U^{--}_m] = \sqrt{2}U_{n+m} + \frac{2n}{3}(c - c')\, \delta_{n+m,0}, \\
[U^{\pm\pm}_n, U^{\pm\pm}_m] &= 0 \\
[U^{\pm\pm}_n, j^{\mp}_{a,r}] &= \sqrt{2}\, j^{\pm}_{a,n+r}, \qquad [U^{\pm\pm}_n, j^{\pm}_{a,r}] = 0, \qquad [U^{\pm\pm}_n, j_m] = 0 \\
\{j^{+}_{a,r}, j^{-}_{b,s}\} &= \frac{c}{3} \varepsilon_{ab}\, \delta_{r+s,0}, \qquad \{j^{\pm}_{a,r}, j^{\pm}_{b,s}\} = 0, \qquad [j^{\pm}_{a,r}, j_m] = 0, \\
[j_n, j_m] &= \frac{c}{3n}\, \delta_{n+m,0}
\end{aligned}
\tag{3.10}
$$

The charges C_α, C'_α are related to c, c' by

$$
C_\alpha = c(1 + 4\alpha^2) + 4\alpha c', \qquad C'_\alpha = -(c' + 2\alpha c) \tag{3.11}
$$

Notice that the (anti) commutators depend nontrivially on α, not only through the values of the central charges, but also through some of the structure constants. The algebra (3.10) can be reduced to the so-called small algebra $N = 4$ [6, 19], admitting only one central extension, $C_{\pm 1/2} = 2(c \pm c')$, $\alpha = \pm 1/2$. The generators of this algebra are L_n, $G^{\pm}_{a,r}$, $U_{k,n}$ for $\alpha = 1/2$ and L_n, $G^{\pm}_{a,r}$, U_n, $U^{\pm\pm}_n$ for $\alpha = -1/2$. It can be obtained by formulating the theory in terms of $J_k = D^+ S_k D^- J$ ($k = 1,2,3$) for $\alpha = 1/2$ and in terms of $J^0 = [D^+, D^-]J$ and $J^{\pm\pm} = D^{\pm} D^{\pm} J$ for $\alpha = -1/2$. As we will see later, in the case of super-W algebra, α must be equal to 1/2 or to $\alpha^{\pm} \neq (1/2, -1/2)$. Then the theory is generated by the superfield J_k or $J^{\pm}_a$. The infinitesimal transformation of J_k is given by

$$\delta_E J_k(Z) = \left(E\partial_z + \frac{1}{2}(D^{+a}E)D_a^- - \frac{1}{2}(D^{-a}E)D_a^+ + \partial_z E\right)J_k(Z) + \frac{(-1)^k}{2}\varepsilon^{kij}(D^+S_jD^-)J_i + \frac{c}{24}(D^+S_kD^-)\partial_z E \tag{3.12}$$

This lead to the following SOPE:

$$\begin{aligned} J_k(Z_1)J_i(Z_2) = {} & \frac{(\theta_{12}^+\ S_k\ \theta_{12}^-)}{Z_{12}}\partial_{z_2} J_i(Z_2) + \frac{1}{2}\frac{\theta_{12}^+\ S_k}{Z_{12}} D_2^- J_i(Z_2) \\ & - \frac{1}{2}\frac{\theta_{12}^- S_k}{Z_{12}} D_2^+ J_i(Z_2) - (-1)^j \frac{\varepsilon^{kij}\ J_j(Z_2)}{Z_{12}} \\ & + \frac{(\theta_{12}^+\ S_{(k}\ \theta_{12}^-)}{Z_{12}^2} J_{i)}(Z_2) - (-1)^{k+j}\delta_{ki}\frac{(\theta_{12}^+\ S_j\ \theta_{12}^-)}{Z_{12}^2} J_j(Z_2) \\ & \frac{1}{2}\frac{(\theta_{12}^+\ \theta_{12}^-}{Z_{12}^2}[\theta_{12)}^+\ S_k\ D_2^-\ J_i(Z_2) + \theta_{12}^-\ S_k\ D_2^+\ J_i(Z_2)] \\ & - \frac{(-1)^j}{2}\frac{(\theta_{12}^+)^2(\theta_{12}^-)^2}{Z_{12}^2}\varepsilon^{kij}J_j(Z_2) \\ & - \frac{c}{12}\frac{(-1)^k\ \delta_{ki}}{Z_{12}^2} + \frac{c}{6}\frac{(-1)^j\ \varepsilon^{kij}\ (\theta_{12}^+\ S_j\ \theta_{12}^-)}{Z_{12}^3} \\ & - \frac{c}{3}(-1)^k\ \delta_{ki}\frac{(\theta_{12}^+)^2\ (\theta_{12}^-)^2}{Z_{12}^4} \end{aligned} \tag{3.13}$$

with $c = c_{1/2}$.

4. THE GENERATORS OF THE $N = 4$ SUPER-W ALGEBRA

The $N = 4$ super-W algebra we will consider contains in addition to the stress-tensor superfield J the primary superfields $\Phi^q_{\pm\nu}$ ($\nu = 0, \ldots, \mu$ for μ integer, $\nu = 1/2, \ldots, \mu$ for μ half-integer, and $\mu = 0, 1/2, \ldots, \Delta$) of conformal weight Δ and Cartan–Weyl charge q. As we will see later in this section, the parameter α, Eq. 3.11, must be equal to $\alpha^\pm(\Delta, \mu, q)$ and show that $\alpha \neq \pm 1/2$ for $\mu \neq \Delta$ and $\alpha = 1/2$ for $\mu = \Delta$. This means that we have two types of $N = 4$ super-W-algebra. For this purpose let us consider the SOPE between the strees-tensor superfield $J(Z)$ and the primary superfield $\Phi^q = \Phi^q_{\pm\nu}$,

$$J(Z_1)\Phi^q(Z_2) = \left[\frac{1}{4}\frac{(\theta_{12}^+)^2(\theta_{12}^-)^2}{Z_{12}}\,\partial_{Z_2} - \frac{1}{2}\frac{(\theta_{12}^+\ \theta_{12}^-)}{Z_{12}}\,(\theta_{12}^{+a}D_{a2}^- + \theta_{12}^{-a}D_{a2}^+) + \frac{\Delta}{4}\frac{(\theta_{12}^+)^2(\theta_{12}^-)^2}{Z_{12}^2} - \frac{q}{4}\frac{(\theta_{12}^+\theta_{12}^-)}{Z_{12}}\right]\Phi^q(Z_2) + \frac{1}{2}\frac{(\theta_{12}^+S_k\theta_{12}^-)}{Z_{12}}\,\Phi_k^q(Z_2) \tag{4.1}$$

where the superfields Φ_k^q ($k = 1, 2, 3$) are given by

$$\begin{aligned}
\Phi_1^q &= \Phi_{\pm\nu,1}^q = \nu\,\Phi_{\mp\nu}^q \\
\Phi_2^q &= \Phi_{\pm\nu,2}^q = \frac{1}{2}\,[(\mu + \nu + 1)\Phi_{\mp(\nu+1)}^q - (\mu - \nu + 1)\Phi_{\mp(\nu-1)}^q] \\
\Phi_3^q &= \Phi\Phi_{\pm\nu,3}^q = -\frac{1}{2}\,[(\mu + \nu + 1)\Phi_{\pm(\nu+1)}^q + (\mu - \nu + 1)\Phi_{\pm(\nu-1)}^q]
\end{aligned} \tag{4.2}$$

The primary superfields $\Phi_{\pm\nu}^q$ are defined as follows:

$$\Phi_{\pm\nu}^q = \Phi_{\mu+\nu,\mu-\nu}^q \pm \Phi_{\mu-\nu,\mu+\nu}^q \tag{4.3.a}$$

where $\Phi_{\mu+\nu,\mu-\nu}^q$ and $\Phi_{\mu-\nu,\mu+\nu}^q$ are sums of all superfields $\Phi_{i_1\ldots i_r j_1\ldots j_s}^q$, Eq. (2.20) with $\mu + \nu$ indices 1, $\mu - \nu$ indices 2, and $\mu - \nu$ indices 1, $\mu + \nu$ indices 2, respectively. Notice that we have

$$\Phi_{+0}^q = \Phi_0^q, \qquad \Phi_{-0}^q = 0 = \Phi_{\pm(\nu+1)}^q \tag{4.3.b}$$

In terms of components the superfields $J(Z)$ and $\Phi^q(Z)$ are written as follows:

$$\begin{aligned}
J(Z) &= \frac{1}{2}\,j(z) + \frac{1}{2}\,\theta^+j^-(z) + \frac{1}{2}\,\theta^-j^+(z) - \frac{1}{4\sqrt{2}}\,(\theta^+)^2U^{--}(z) \\
&\quad + \frac{1}{4\sqrt{2}}\,(\theta^-)^2U^{++}(z) - \frac{1}{4}\,(\theta^+\theta^-)U(z) \\
&\quad - \frac{(-1)^k}{4}\,(\theta^+S_k\theta^-)U_k(z) + \frac{1}{2}\,(\theta^+\theta^-)(\theta^+K^-(z) + \theta^-K^+(z)) \\
&\quad + \frac{1}{4}\,(\theta^+)^2(\theta^-)^2t(z)
\end{aligned} \tag{4.4a}$$

$$\begin{aligned}
\Phi^q(Z) &= \phi^q(z) + \theta^+\phi^{q-1} + \theta^-\phi^{q+1} - \frac{1}{2\sqrt{2}}\,(\theta^+)^2H^{q-2}(z) \\
&\quad + \frac{1}{2\sqrt{2}}\,(\theta^-)^2H^{q+2}(z) - \frac{1}{2}\,(\theta^+\theta^-)H^q(z)
\end{aligned}$$

$$- \frac{(-1)^k}{2} (\theta^+ S_k \theta^-) R^q_k(z) + (\theta^+\theta^-)(\theta^+ R^{q-1} + \theta^- R^{q+1})$$

$$+ \frac{1}{2} (\theta^+)^2 (\theta^-)^2 w^q(z) \tag{4.4b}$$

Now we analyze the conditions for which the following fields are primary:

$$\begin{aligned} W^q &= w^q + a_1 \partial_z H^q + a_2 \partial_z R^q_{k,k} + a_3 \partial_z^2 \phi^q \\ F^{q\pm1}_a &= R^{q\pm1}_a + b_1 \partial_z \phi^{q\pm1}_a + b_2 (-1)^k (S)_{ab} \partial_z \phi^{q\pm1b}_k \\ E^q_k &= R^q_k + b_3(-1)^k \partial_z \phi^q_k, \qquad B^q = H^q + b_4 \partial_z \phi^q \end{aligned} \tag{4.5a}$$

In fact the fields w^q, $R^{q\pm1}_a$, R^q_k, and H^q are not primary in the sense of Virasoro algebra in general. By using (4.1), we find that

$$\begin{aligned} T(z_1)\phi^q(z_2) &= \frac{\Delta}{z_{12}^2} \phi^q(z_2) + \frac{1}{z_{12}} \partial_{z_2} \phi^q(z_2) \\ T(z_1)\phi^{q\pm1}_a(z_2) &= \frac{\Delta + 1/2}{z_{12}^2} \phi^{q\pm1}_a(z_2) + \frac{1}{z_{12}} \partial_{z_2} \phi^{q\pm1}_a(z_2) \\ T(z_1)H^q(z_2) &= \frac{\Delta + 1}{z_{12}^2} H^q(z_2) + \frac{1}{z_{12}} \partial_{z_2} H^q(z_2) - q \frac{1 - 2\alpha}{z_{12}^3} \phi^q(z_2) \\ T(z_1)R^q_k(z_2) &= \frac{\Delta + 1}{z_{12}^2} R^q_k(z_2) + \frac{1}{z_{12}} \partial_{z_2} R^q_k(z_2) - \frac{2\,(1 + 2\alpha}{z_{12}^3} (-1)^k \phi^q_k(z_2) \\ T(z_1)R^{q\pm1}_a(z_2) &= \frac{\Delta + 3/2}{z_{12}^2} R^{q\pm1}_a(z_2) + \frac{1}{z_{12}} \partial_{z_2} R^{q\pm1}_a(z_2) \\ &\quad + \frac{(q/2)(1 - 2\alpha) \mp 2\alpha}{z_{12}^3} \phi^{q\pm1}_a(z_2) \\ &\quad \pm \frac{1 + 2\alpha}{z_{12}^3} (-1)^k (S)_{ab} \phi^{q\pm1b}_k(z_2) \\ T(z_1)w^q(z_2) &= \frac{\Delta + 2}{z_{12}^2} w^q(z_2) + \frac{1}{z_{12}} \partial_{z_2} w^q(z_2) \\ &\quad + \frac{q}{4} \frac{1 - 2\alpha}{z_{12}^3} H^q(z_2) + \frac{1}{2} \frac{1 + 2\alpha}{z_{12}^3} R^q_k(z_2) \\ &\quad - \frac{2\alpha}{z_{12}^3} \partial_{z_2} \phi^q(z_2) + \frac{6\alpha\Delta}{z_{12}^4} \phi^q(z_2) \end{aligned} \tag{4.5b}$$

where $T(z) = t(z) + \alpha\partial_z^2 j(z)$. From the expressions of the OPEs $T(z_1)W^q(z_2)$, $T(z_1)F_a^{q\pm1}(z_2)$, $T(z_1)E_k^q(z_2)$, and $T(z_1)B^q(z_2)$, which can be easily deduced by using the expressions (4.5a), and (4.5b), we can give the conditions for which the fields (4.5a) are primary:

$$\alpha = \alpha^{\pm} = \frac{8\Delta(\Delta + 1) - 4\mu(\mu + 1) - q^2 \pm 4\sqrt{(\Delta(\Delta + 1) - \mu(\mu + 1))(4\Delta(\Delta + 1) - q^2)}}{2(4\mu(\mu + 1) - q^2)}$$

$$a_1 = a_1^{\pm} = -q\,\frac{1 - 2\alpha^{\pm}}{8\Delta}, \qquad a_2 = a_2^{\pm} = \frac{1 + 2\alpha^{\pm}}{4(\Delta + 1)}, \qquad a_3 = a_3^{\pm} = \alpha^{\pm}$$

$$b_1 = b_1^{\pm} = -\frac{q(1 - 2\alpha^{\pm}) \mp 4\alpha^{\pm}}{2(2\Delta + 1)}, \qquad b_2 = b_2^{\pm} = \mp\frac{1 + 2\alpha^{\pm}}{2\Delta + 1}$$

$$b_3 = b_3^{\pm} = \frac{1 + 2\alpha^{\pm}}{\Delta}, \qquad b_4 = b_4^{\pm} = \frac{q(1 - 2\alpha^{\pm})}{2\Delta} \tag{4.6a}$$

So we can see that for $\mu \neq \Delta$ we have $\alpha \neq \pm 1/2$ and for $\mu = \Delta$ we have

$$\begin{aligned} &\alpha = \frac{1}{2}, \qquad a_1 = 0, \qquad a_2 = -\frac{1}{2(\Delta + 1)}, \qquad a_3 = \alpha = \frac{1}{2} \\ &b_1 = \pm\frac{1}{2\Delta + 1}, \qquad b_2 = \mp\frac{2}{2\Delta + 1}, \qquad b_3 = \frac{2}{\Delta}, \qquad b_4 = 0 \end{aligned} \tag{4.6b}$$

Conseqently for $\mu \neq \Delta$ $(\alpha = \alpha^{\pm} \neq (1/2, -1/2))$ the $N = 4$ super-W algebra admits as subalgebra the large $N = 4$ superconformal algebra, which is generated by

$$T(z) = t(z) + \alpha^{\pm}\partial_z^2 j(z), \qquad G_a^{\pm}(z) = K_a^{\pm}(z) \pm 2\alpha^{\pm}\partial_z j_a^{\pm}(z)$$
$$U_k(z), \quad U(z), \quad U^{\pm\pm}(z), \quad j_a^{\pm}(z), \quad j(z)$$

For $\mu = \Delta$ $(\alpha = 1/2)$ the $N = 4$ super-W algebra admits as subalgebra the small $N = 4$ superconformal algebra, which is generated by

$$T(z) = t(z) + \tfrac{1}{2}\partial_z^2 j(z), \qquad G_a^{\pm}(z) = K_a^{\pm}(z) \pm \partial_z j_a^{\pm}(z), \qquad U_k\,(z)$$

Let us note that if we take the components of the superfield $\Phi^q_{i_1\ldots i_r j_1\ldots j_s}$ and we analyze their redefinitions, $W^q_{i_1\ldots i_r j_1\ldots j_s}$, $F^{q\pm1}_{a,i_1\ldots i_r j_1\ldots j_s}$, $E^q_{k,i_1\ldots i_r j_1\ldots j_s}$, and $B^q_{i_1\ldots i_r j_1\ldots j_s}$, which are similar to (4.5a), we deduce that the field $W^q_{i_1\ldots i_r j_1\ldots j_s}$ is not primary for any value of a_K $(K = 1, 2, 3)$ and α. We conclude that in order to obtain primary fields we have to take either the superfields (4.3a) or the following superfields:

$$\Psi^q_{\pm\sigma} = \Psi^q_{\mu+\sigma,\mu-\sigma} \pm \Psi^q_{\mu-\sigma,\mu+\sigma}$$

$$(\sigma = 0, 1/2, \ldots, \rho;\ \rho = 0, 1/2, \ldots, \mu - 1;\ \mu = 0,1/2, \ldots, \Delta) \quad (4.7)$$

where σ and ρ are integer if μ is integer and half-integer if μ is half-integer. Here $\Psi^q_{\rho+\sigma,\rho-\sigma}$ and $\Psi^q_{\rho-\sigma,\rho+\sigma}$ are certain combinations of tensor superfields $\Phi^q_{i_1\ldots i_r j_1\ldots j_s}$ with $\mu + \sigma$ indices 1, $\mu - \sigma$ indices 2, and $\mu - \sigma$ indices 1, $\mu + \sigma$ indices 2, respectively. The superfields $\Psi^q_{\pm\sigma}$ give similar results.

As we will see later, the superfields $\Phi^q_{\pm\nu}$ ($\mu \neq \Delta$) are null superfields for $q \neq 0$. We also remark for the second type ($\mu = \Delta$) the absence of $U(1)$ symmetry. For these reason, we restrict consideration to $q = 0$ in the remainder of this paper. In terms of components the expressions (4.1) takes the following form:

$$T(z_1)\phi(z_2) = \frac{\Delta}{z_{12}^2}\,\phi(z_2) + \frac{1}{z_{12}}\,\partial_{z_2}\phi(z_2)$$

$$T(z_1)\phi_a^{\pm}(z_2) = \frac{\Delta + 1/2}{z_{12}^2}\,\phi_a^{\pm}(z_2) + \frac{1}{z_{12}}\,\partial_{z_2}\phi_a^{\pm}(z_2)$$

$$T(z_1)H^{\pm\pm}(z_2) = \frac{\Delta + 1}{z_{12}^2}\,H^{\pm\pm}(z_2) + \frac{1}{z_{12}}\,\partial_{z_2}H^{\pm\pm}(z_2)$$

$$T(z_1)H(z_2) = \frac{\Delta + 1}{z_{12}^2}\,H(z_2) + \frac{1}{z_{12}}\,\partial_{z_2}H(z_2)$$

$$T(z_1)E_k(z_2) = \frac{\Delta + 1}{z_{12}^2}\,E_k(z_2) + \frac{1}{z_{12}}\,\partial_{z_2}E_k(z_2)$$

$$T(z_1)F_a^{\pm}(z_2) = \frac{\Delta + 3/2}{z_{12}^2}\,F_a^{\pm}(z_2) + \frac{1}{z_{12}}\,\partial_{z_2}F_a^{\pm}(z_2)$$

$$T(z_1)W(z_2) = \frac{\Delta + 2}{z_{12}^2}\,W(z_2) + \frac{1}{z_{12}}\,\partial_{z_2}W(z_2)$$

$$G_a^{\pm}(z_1)\phi(z_2) = +\frac{1}{z_{12}}\,\phi_a^{\pm}(z_2)$$

$$G_a^{\pm}(z_1)\phi_b^{\pm}(z_2) = \frac{-1/\sqrt{2}}{z_{12}}\,\varepsilon_{ab}H^{\pm\pm}(z_2)$$

$$G_a^{\pm}(z_1)\phi_b^{\mp}(z_2) = \frac{\pm 1/2}{z_{12}}\,\varepsilon_{ab}H(z_2) - \frac{1/2}{z_{12}}\,(S_k)_{ab}E_k(z_2) - \frac{\varepsilon_{ab}}{z_{12}}\,\partial_{z_2}\phi(z_2)$$

$$+\frac{1+2\alpha}{2\Delta}\frac{(-1)^k}{z_{12}}(S_k)_{ab}\partial_{z_2}\phi_k(z_2)-\frac{2\Delta}{z_{12}^2}\varepsilon_{ab}\phi(z_2)$$

$$+\frac{(1+2\alpha)(-1)^k}{z_{12}^2}(S_k)_{ab}\phi_k(z_2)$$

$$G_a^{\pm}(z_1)H^{\pm\pm}(z_2)=\text{regular}$$

$$G_a^{\pm}(z_1)H^{\mp\mp}(z_2)=\frac{\sqrt{2}}{z_{12}}F_a^{\mp}(z_2)\pm\frac{2\Delta+2\alpha+1}{2\Delta+1}\frac{1}{z_{12}}\partial_{z_2}\phi_a^{\mp}(z_2)$$

$$\mp\frac{1+2\alpha}{2\Delta+1}\frac{(-1)^k}{z_{12}}(S_k)_{ab}\partial_{z_2}\phi_k^{\mp b}(z_2)$$

$$\pm(2\Delta+2\alpha+1)\frac{\sqrt{2}}{z_{12}^2}\phi_a^{\mp}(z_2)$$

$$\mp(1+2\alpha)\frac{\sqrt{2}(-1)^k}{z_{12}}(S_k)_{ab}\partial_{z_2}\phi_k^{\mp b}(z_2)$$

$$G_a^{\pm}(z_1)H(z_2)=\frac{\mp 1}{z_{12}}F_a^{\pm}(z_2)+\frac{2\Delta+2\alpha+1}{2\Delta+1}\frac{1}{z_{12}}\partial_{z_2}\phi_a^{\pm}(z_2)$$

$$-\frac{1+2\alpha}{2\Delta+1}\frac{(-1)^k}{z_{12}}(S_k)_{ab}\partial_{z_2}\phi_k^{\pm b}(z_2)$$

$$+(2\Delta+2\alpha+1)\frac{1}{z_{12}^2}\phi_a^{\pm}(z_2)$$

$$\mp(1+2\alpha)\frac{(-1)^k}{z_{12}}(S_k)_{ab}\partial_{z_2}\phi_k^{\pm b}(z_2)$$

$$G_a^{\pm}(z_1)E_k(z_2)=\frac{(-1)^k}{z_{12}}(S_k)_{ab}F^{\pm b}(z_2)\pm\frac{2\Delta-2\alpha+1}{2\Delta+1}\frac{(-1)^k}{z_{12}}(S_k)_{ab}\partial_{z_2}\phi^{\pm b}(z_2)$$

$$\pm\frac{\Delta+1}{\Delta(2\Delta+1)}\frac{(1+2\alpha)(-1)^k}{z_{12}}\partial_{z_2}\phi_{a,k}^{\pm}(z_2)$$

$$\mp\frac{1+2\alpha}{2\Delta+1}\frac{\varepsilon^{kij}}{z_{12}}(S_j)_{ab}\partial_{z_2}\phi_i^{\pm b}(z_2)$$

$$\pm(2\Delta-2\alpha+1)\frac{(-1)^k}{z_{12}^2}(S_k)_{ab}\phi^{\pm b}(z_2)$$

$$\pm \frac{\Delta + 1}{\Delta} \frac{(1 + 2\alpha)(-1)^k}{z_{12}^2} \phi^{\pm}_{a,k}(z_2)$$

$$\mp (1 + 2\alpha) \frac{\varepsilon^{kij}}{z_{12}^2} (S_j)_{ab} \phi_i^{\pm b}(z_2)$$

$$G_a^{\pm}(z_1) F_b^{\pm}(z_2) = \pm \frac{2\Delta - 2\alpha + 1}{\sqrt{2}(2\Delta + 1)} \frac{\varepsilon_{ab}}{z_{12}} \partial_{z_2} H^{\pm\pm}(z_2)$$

$$\mp \frac{2\alpha + 1}{\sqrt{2}(2\Delta + 1)} \frac{(-1)^k}{z_{12}} (S_k)_{ab} \partial_{z_2} H_k^{\pm\pm}(z_2)$$

$$\pm 2(\Delta + 1) \frac{2\Delta - 2\alpha + 1}{\sqrt{2}(2\Delta + 1)} \frac{\varepsilon_{ab}}{z_{12}^2} H^{\pm\pm}(z_2)$$

$$\mp 2(\Delta + 1) \frac{2\alpha + 1)}{\sqrt{2}(2\Delta + 1)} \frac{(-1)^k}{z_{12}^2} (S_k)_{ab} H_k^{\pm\pm}(z_2)$$

$$G_a^{\pm}(z_1) F_b^{\mp}(z_2) = \pm \frac{\varepsilon_{ab}}{z_{12}} W(z_2) + \frac{2\Delta - 2\alpha + 1}{2(2\Delta + 1)} \frac{\varepsilon_{ab}}{z_{12}} \partial_{z_2} H(z_2)$$

$$\pm \frac{\Delta(2\alpha + 1)}{2(2\Delta + 1)(\Delta + 1)} \frac{\varepsilon_{ab}}{z_{12}} \partial_{z_2} E_{k,k}(z_2)$$

$$- \frac{2\alpha + 1}{2(2\Delta + 1)} \frac{(-1)^k}{z_{12}} (S_k)_{ab} \partial_{z_2} H_k(z_2)$$

$$\pm \frac{2\Delta + 2\alpha + 1}{2(2\Delta + 1)} \frac{(S_k)_{ab}}{z_{12}} \partial_{z_2} E_k(z_2)$$

$$\mp \frac{2\alpha + 1}{2(2\Delta + 1)} \frac{(-1)^k \varepsilon^{kij}}{z_{12}} (S_j)_{ab} \partial_{z_2} E_{k,i}(z_2)$$

$$+ \frac{(\Delta + 1)(2\Delta - 2\alpha + 1)}{2\Delta + 1} \frac{\varepsilon_{ab}}{z_{12}^2} H(z_2)$$

$$\pm \frac{(\Delta + 1)(2\Delta + 2\alpha + 1)}{2\Delta + 1} \frac{(S_k)_{ab}}{z_{12}^2} E_k(z_2)$$

$$- \frac{(\Delta + 1)(2\alpha + 1)}{2\Delta + 1} \frac{(-1)^k}{z_{12}^2} (S_k)_{ab} H_k(z_2)$$

$$\pm \frac{\Delta(2\alpha + 1)}{2\Delta + 1} \frac{\varepsilon_{ab}}{z_{12}^2} E_{k,k}(z_2)$$

$$\mp \frac{8\alpha\Delta(\Delta + 1) - \mu(\mu + 1)(2\alpha + 1)^2}{2(2\Delta + 1)(\Delta + 1)} \frac{\varepsilon_{ab}}{z_{12}} \partial_{z_2}^2 \phi(z_2)$$

$$\pm \frac{2(8\alpha\Delta(\Delta + 1) - \mu(\mu + 1)(2\alpha + 1)^2)}{2\Delta + 1} \frac{\varepsilon_{ab}}{z_{12}^3} \phi(z_2)$$

$$G_a^{\pm}(z_1)W(z_2) = \frac{1/2}{z_{12}} \partial_{z_2} F_a^{\pm}(z_2) - \frac{2\alpha + 1}{4(\Delta + 1)} \frac{(-1)^k}{z_{12}} (S_k)_{ab} \partial_{z_2} F_k^{\pm b}(z_2)$$

$$+ \frac{1}{2} \frac{2\Delta + 3}{z_{12}^2} F_a^{\pm}(z_2)$$

$$- \frac{2\Delta + 3}{4(\Delta + 1)} \frac{(2\alpha + 1)(-1)^k}{z_{12}^2} (S_k)_{ab} F_k^{\pm \mathrm{b}}(z_2)$$

$$\pm \frac{8\Delta(\Delta + 1) - \mu(\mu + 1)(2\alpha + 1)^2}{4(\Delta + 1)(2\Delta + 1)} \frac{1}{z_{12}} \partial_{z_2}^2 \phi_a^{\pm}(z_2)$$

$$\pm \frac{8\Delta(\Delta + 1) - \mu(\mu + 1)(2\alpha + 1)^2}{2(\Delta + 1)(2\Delta + 1)} \frac{1}{z_{12}^2} \partial_{z_2} \phi_a^{\pm}(z_2)$$

$$\mp \frac{8\Delta(\Delta + 1) - \mu(\mu + 1)(2\alpha + 1)^2}{2(\Delta + 1)} \frac{1}{z_{12}^3} \phi_a^{\pm}(z_2)$$

$$U_k(z_1)\phi(z_2) = -\frac{2(-1)^k}{z_{12}} \phi_k(z_2)$$

$$U_k(z_1)\phi_a^{\pm}(z_2) = \frac{-2(-1)^k}{z_{12}} \phi_{a,k}^{\pm}(z_2) - \frac{2(-1)^k}{z_{12}} (S_k)_{ab} \phi^{\pm b}(z_2)$$

$$U_k(z_1)H^{\pm\pm}(z_2) = -\frac{2(-1)^k}{z_{12}} H_k^{\pm\pm}(z_2)$$

$$U_k(z_1)H(z_2) = -\frac{2(-1)^k}{z_{12}} H_k(z_2)$$

$$U_k(z_1)E_i(z_2) = -\frac{2(-1)^k}{z_{12}} E_{i,k}(z_2) - 2(-1)^j \frac{\varepsilon^{kij}}{z_{12}} E_j(z_2)$$

$$- \frac{4\Delta(-1)^k}{z_{12}^2} \delta_{ki} \phi(z_2)$$

$$- \frac{2(2\alpha + 1)}{\Delta} \frac{2(-1)^{k+i}}{z_{12}^2} \phi_{k,i}(z_2) + 4 \frac{\varepsilon^{kij}}{z_{12}^2} \phi_j(z_2)$$

$$
\begin{aligned}
U_k(z_1)F_a^{\pm}(z_2) &= -\frac{2(-1)^k}{z_{12}}\,F_{a,k}^{\pm}(z_2) - \frac{(-1)^k}{z_{12}}\,(S_k)_{ab}F^{\pm b}(z_2)\\
&\mp \frac{6\alpha+1}{2\Delta+1}\,\frac{(-1)^k}{z_{12}^2}\,\phi_{a,k}^{\pm}(z_2)\\
&\pm \frac{(2\Delta+1)^2-2\alpha}{2\Delta+1}\,\frac{(-1)^k}{z_{12}^2}\,(S_k)_{ab}\phi^{\pm b}(z_2)\\
&\pm \frac{2(2\alpha+1)}{2\Delta+1}\,\frac{(-1)^{k+i}}{z_{12}^2}\,(S_i)_{ab}\phi_{k,i}^{\pm b}(z_2)\\
&\pm \frac{4\Delta+2\alpha+3}{2\Delta+1}\,\frac{\varepsilon^{kij}}{z_{12}^2}\,(S_j)_{ab}\phi_i^{\pm b}(z_2)\\
U_k(z_1)W(z_2) &= -\frac{2(-1)^k}{z_{12}}\,W(z_2) + \frac{\Delta+1}{z_{12}^2}\,E_k(z_2)\\
&+ \frac{2\alpha+1}{2(\Delta+1)}\,\frac{1}{z_{12}^2}\,E_{i,i,k}(z_2)\\
&- \frac{(-1)^i\varepsilon^{kij}}{z_{12}^2}\,E_{i,j}(z_2)\\
&- \frac{8\Delta(\Delta+1)-\mu(\mu+1)(2\alpha+1)^2}{2\Delta(\Delta+1)}\,\frac{(-1)^k}{z_{12}^2}\,\partial_{z_2}\phi_k(z_2)
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
U(z_1)\phi(z_2) &= \text{regular}\\
U(z_1)\phi_a^{\pm}(z_2) &= \pm\frac{1}{z_{12}}\,\phi_a^{\pm}(z_2)\\
U(z_1)E_k(z_2) &= \text{regular}\\
U(z_1)H(z_2) &= -\frac{4\Delta}{z_{12}}\,\phi(z_2)\\
U(z_1)H^{\pm\pm}(z_2) &= \pm\frac{2}{z_{12}}\,H^{\pm\pm}(z_2)\\
U(z_1)F_a^{\pm}(z_2) &= \pm\frac{1}{z_{12}}\,F_a^{\pm}(z_2) + \frac{(2\Delta+1)^2+2\alpha}{2\Delta+1}\,\frac{1}{z_{12}^2}\,\phi_a^{\pm}(z_2)
\end{aligned}
$$

$$- \frac{2\alpha + 1}{2\Delta + 1}\frac{(-1)^k}{z_{12}^2}(S_k)_{ab}\phi^{\pm b}(z_2)$$

$$U(z_1)W(z_2) = -\frac{\Delta + 1}{z_{12}^2} H(z_2)$$

$$U^{\pm\pm}(z_1)\phi(z_2) = \text{regular}$$

$$U^{\pm\pm}(z_1)\phi_a^{\pm}(z_2) = \text{regular}$$

$$U^{\pm\pm}(z_1)H^{\pm\pm}(z_2) = \text{regular}$$

$$U^{\pm\pm}(z_1)\phi_a^{\mp}(z_2) = \frac{\sqrt{2}}{z_{12}}\phi_a^{\pm}(z_2)$$

$$U^{\pm\pm}(z_1)E_k(z_2) = \text{regular}$$

$$U^{\pm\pm}(z_1)F_a^{\pm}(z_2) = \text{regular}$$

$$U^{\pm\pm}(z_1)H(z_2) = \mp\frac{2}{z_{12}} H^{\pm\pm}(z_2)$$

$$U^{\pm\pm}(z_1)H^{\mp\mp}(z_2) = \pm\frac{2}{z_{12}} H(z_2) - \frac{4\Delta}{z_{12}^2}\phi(z_2)$$

$$U^{\pm\pm}(z_1)F_a^{\mp}(z_2) = -\frac{\sqrt{2}\, F_a^{\pm}}{z_{12}F(z_2)} \mp \frac{(2\Delta + 1)^2 + 2\alpha}{2\Delta + 1}\frac{\sqrt{2}}{z_{12}^2}\phi_a^{\pm}(z_2)$$
$$- \frac{2\alpha + 1}{(2\Delta + 1)}\frac{\sqrt{2}(-1)^k}{z_{12}^2}(S_k)_{ab}\phi_k^{\pm \mathrm{b}}(z_2)$$

$$U^{\pm\pm}(z_1)W(z_2) = -\frac{\Delta + 1}{z_{12}^2} H^{\pm\pm}(z_2)$$

$$j_a^{\pm}(z_1)\phi(z_2) = \text{regular}$$

$$j_a^{\pm}(z_1)\phi_b^{\pm}(z_2) = \text{regular}$$

$$j_a^{\pm}(z_1)\phi_b^{\mp}(z_2) = \mp\frac{(-1)^k}{z_{12}}(S_k)_{ab}\,\phi_k(z_2)$$

$$j_a^{\pm}(z_1)H^{\pm\pm}(z_2) = \text{regular}$$

$$j_a^{\pm}(z_1)H^{\mp\mp}(z_2) = -\frac{\sqrt{2}}{z_{12}}\,\phi_a^{\mp}(z_2) + \frac{\sqrt{2}(-1)^k}{z_{12}}\,(S_k)_{ab}\phi_k^{\mp b}(z_2)$$

$$j_a^{\pm}(z_1)H(z_2) = \mp\frac{1}{z_{12}}\,\phi_a^{\pm}(z_2) \pm \frac{(-1)^k}{z_{12}}\,(S_k)_{ab}\phi_k^{\pm b}(z_2)$$

$$j_a^{\pm}(z_1)E_k(z_2) = \frac{(-1)^k}{z_{12}}\,(S_k)_{ab}\phi_k^{\pm b}(z_2)$$

$$-\frac{(-1)^k}{z_{12}}\,\phi_{a,k}^{\pm}(z_2) + \frac{\varepsilon^{kij}}{z_{12}}\,(S_j)_{ab}\phi_i^{\pm b}(z_2)$$

$$j_a^{\pm}(z_1)F_b^{\pm}(z_2) = \frac{1/\sqrt{2}}{z_{12}}\,\varepsilon_{ab}H^{\pm\pm}(z_2) + \frac{1/\sqrt{2}(-1)^k}{z_{12}}\,(S_k)_{ab}H_k^{\pm\pm}(z_2)$$

$$j_a^{\pm}(z_1)F_b^{\mp}(z_2) = \pm\frac{1/2}{z_{12}}\,\varepsilon_{ab}H(z_2) - \frac{1/2}{z_{12}}\,(S_k)_{ab}E_k(z_2)$$

$$\pm\frac{1/2(-1)^k}{z_{12}}\,(S_k)_{ab}H_k(z_2)$$

$$-\frac{1/2}{z_{12}}\,\varepsilon_{ab}E_{k,k}(z_2) + \frac{1/2(-1)^k\varepsilon^{kij}}{z_{12}}\,(S_j)_{ab}E_{k,i}(z_2)$$

$$-\frac{1}{2\Delta+1}\frac{(-1)^k}{z_{12}}\,(S_k)_{ab}\partial_{z_2}\phi_k(z_2)$$

$$+\frac{2\Delta(2\Delta+1)-\mu(\mu+1)(2\alpha+1)}{2\Delta(2\Delta+1)}\frac{\varepsilon_{ab}}{z_{12}}\,\partial_{z_2}\phi(z_2)$$

$$+\frac{2\Delta}{2\Delta+1}\frac{(-1)^k}{z_{12}^2}\,(S_k)_{ab}\phi_k(z_2)$$

$$j_a^{\pm}(z_1)W(z_2) = \pm\frac{1/2(-1)^k}{z_{12}}\,(S_k)_{ab}F_k^{\pm b}(z_2)$$

$$+\frac{1}{2(2\Delta+1)}\frac{(-1)^k}{z_{12}}\,(S_k)_{ab}\partial_{z_2}\phi_k^{\pm b}(z_2)$$

$$-\frac{2(\Delta+1)(2\Delta+1)-\mu\,(\mu+1)(2\alpha+1)}{2(\Delta+1)(2\Delta+1)}$$

$$
\begin{aligned}
&\times \frac{1}{z_{12}}\,\partial_{z_2}\phi_a^{\pm}(z_2)\\
&- \frac{(-1)^k}{z_{12}^2}\,(S_k)_{ab}\phi_k^{\pm b}(z_2)\\
&+ \frac{2(\Delta+1)(2\Delta+1)-\mu(\mu+1)(2\alpha+1)}{4(\Delta+1)}\\
&\times \frac{1}{z_{12}^2}\,\phi_a^{\pm}(z_2)
\end{aligned}
$$

$$
\begin{aligned}
j(z_1)\phi(z_2) &= \text{regular}\\
j(z_1)\phi_b^{\pm}(z_2) &= \text{regular}\\
j(z_1)H^{\pm\pm}(z_2) &= \text{regular}\\
j(z_1)H(z_2) &= \text{regular}\\
j(z_1)E_k(z_2) &= -\frac{2(-1)^k}{z_{12}}\,\phi_k(z_2)\\
j(z_1)F_a^{\pm}(z_2) &= \mp\frac{1}{z_{12}}\,\phi_a^{\pm}(z_2) \pm \frac{(-1)^k}{z_{12}}\,(S_k)_{ab}\phi_k^{\pm b}(z_2)\\
j(z_1)W(z_2) &= \frac{1/2}{z_{12}}\,E_{k,k}(z_2)\\
&\quad - \frac{2\Delta(\Delta+1)-\mu(\mu+1)(2\alpha+1)}{2\Delta(\Delta+1)}\,\frac{1}{z_{12}}\,\partial_{z_2}\phi(z_2)\\
&\quad + \frac{2\Delta(\Delta+1)-\mu(\mu+1)(2\alpha+1)}{2(\Delta+1)}\,\frac{1}{z_{12}^2}\,\phi(z_2)
\end{aligned}
\tag{4.9}
$$

with $\alpha = \alpha^{\pm}(\Delta, \mu, q) = 0$, $\mu = 0, \ldots, \Delta$.

We note that for $\mu = \Delta$ ($\alpha = 1/2$) we have simply the OPE between $T(z)$, $G_a^{\pm}(z)$, $U_k(z)$, and the components of the superfields Φ, namely the relations (4.8).

To handle the SOPE $\Phi(Z_1)\Phi(Z_2)$ in a manifestly supersymmetric fashion it is useful to define the $N = 4$ superconformal generators by super-Fourier expansion. These are given for the two situations by

$$\tilde{L}_n(Z_2) = \frac{1}{4}\oint_{C_{z_2}} dZ_1\, Z_{12}^{n+1}[\theta_{12}^{+a} J_a^-(Z_1) - \theta_{12}^{-a} J_a^+(Z_1)] + \alpha n(n+1)\tilde{j}_n(Z_2)$$

$$\tilde{G}^{\pm}_{a,r}(Z_2) = \mp\frac{1}{3}\oint_{C_{z_2}} dZ_1\, Z_{12}^{r+1/2}[2(\theta^{\pm}_{a12}\theta_{12}^{\mp b})J_b^-(Z_1) - (\theta_{12}^{\pm})^2 J_a^{\mp}(Z_1)] \mp 2\alpha(\mathrm{r} + 1/2)\tilde{j}^{\pm}_{a,r}(Z_2)$$

$$\tilde{U}_{k,n}(Z_2) = \oint_{C_{z_2}} dZ_1\, Z_{12}^{n+1}(\theta_{12}^{+} S_k \theta_{12}^{-})[\theta_{12}^{+a} J_a^-(Z_1) - \theta_{12}^{-a} J_a^+(Z_1)]$$

$$\tilde{U}_n(Z_2) = \oint_{C_{z_2}} dZ_1\, Z_{12}^{n+1}(\theta_{12}^{+}\theta_{12}^{-})[\theta_{12}^{+a} J_a^-(Z_1) - \theta_{12}^{-a} J_a^+(Z_1)]$$

$$\tilde{U}_n^{\pm\pm}(Z_2) = \sqrt{2}\oint_{C_{z_2}} dZ_1\, Z_{n12}(\theta_{12}^{+}\theta_{12}^{-})\theta_{12}^{\pm a} J_a^{\pm}(Z_1)$$

$$\tilde{j}^{\pm}_{a,r}(Z_2) = \mp\frac{1}{2}\oint_{C_{z_2}} dZ_1\, Z_{12}^{r-1/2}(\theta_{12}^{+})^2(\theta_{12}^{-})^2 J_a^{\pm}(Z_1)$$

$$\tilde{j}_n(Z_2) = \frac{1}{2n}\oint_{C_{z_2}} dZ_1\, Z_{12}^{n}(\theta_{12}^{+}\theta_{12}^{-})[\theta_{12}^{+a} J_a^-(Z_1) + \theta_{12}^{-a} J_a^+(Z_1)] \tag{4.10a}$$

for $\mu \neq \Delta$ and

$$\tilde{L}_n(Z_2) = -\frac{(-1)^k}{6}\oint_{C_{z_2}} dZ_1\, Z_{12}^{n+1}(\theta_{12}^{+} S_k \theta_{12}^{-})J_k(Z_1)$$

$$\tilde{G}^{\pm}_{a,r}(Z_2) = -\frac{2(-1)^k}{3}\oint_{C_{z_2}} dZ_1\, Z_{12}^{r+1/2}(\theta_{12}^{+} S_k \theta_{12}^{-})\theta^{\pm}_{a12} J_k(Z_1) \tag{4.10b}$$

$$\tilde{U}_{k,n}(Z_2) = \frac{1}{2}\oint_{C_{z_2}} dZ_1\, Z_{12}^{n+1}(\theta_{12}^{+})^2(\theta_{12}^{-})^2 J_k(Z_1)$$

for $\mu = \Delta$. Here $n \in \mathbf{Z}$ and $r \in \mathbf{Y} = \mathbf{Z} + 1/2$ (Neveu–Schwarz sector) or $\mathbf{Z}$ (Ramond sector). Notice that the non-tilde generators (3.9) are obtained by setting $Z_2 = 0$ in the expression of the generators (4.11). On the other hand, we define the $N = 4$ super-primary state by

$$|\phi^q\rangle = \phi^q(0)|0\rangle \tag{4.11a}$$

which satisfies the following relations:

$$L_0|\phi^q\rangle = \Delta|\phi^q\rangle, \qquad U_{k,0}|\phi^q\rangle = -2(-1)^k|\phi^q_k\rangle$$

$$L_n|\phi^q\rangle = 0, \qquad U_{k,n}|\phi^q\rangle = 0, \qquad G^{\pm}_{a,r}|\phi^q\rangle = 0 \quad (n, r > 0) \tag{4.11b}$$

$$U_0|\phi^q\rangle = q|\phi^q\rangle, \qquad U_0^{\pm\pm}|\phi^q\rangle = 0 \tag{4.11c}$$
$$U_n|\phi^q\rangle = 0, \qquad U_n^{\pm\pm}|\phi^q\rangle = 0, \qquad j^{\pm}_{a,r}|\phi^q\rangle = 0, \qquad j_n|\phi^q\rangle = 0 \quad (n, r > 0)$$

For the first type ($\mu = \Delta$) we have only the expressions (4.11b). In fact in this case the $N = 4$ superconformal algebra is generated only by L_n, $G^{\pm}_{a,r}$, and $U_{k,n}$. The other members of the supermultiplet to which $|\phi\rangle$ ($q = 0$) belongs are given by

$$\begin{aligned}
|\phi^{\pm}_a\rangle &= \pm G^{\pm}_{a,-1/2}|\phi\rangle, \; |H^{\pm\pm}\rangle = \mp\frac{\sqrt{2}}{2}\, G^{\pm}_{-1/2}\, G^{\pm}_{-1/2}|\phi\rangle \\
|H\rangle &= [2L_{-1} - G^{\pm}_{-1/2}\, G^{-}_{-1/2}]\phi\rangle \\
|E_k\rangle &= G^{+}_{-1/2}S_k G^{-}_{-1/2}|\phi\rangle + \frac{(2\alpha + 1)(-1)^k}{\Delta}\, L_{-1}|\phi_k\rangle \\
|F^{\pm}_a\rangle &= \left[\pm G^{\pm}_{-1/2}\, G^{\mp}_{-1/2} - \frac{6\Delta - 2\alpha + 3}{2\Delta + 1}\, L_{-1}\right] G^{\pm}_{a,-1/2}|\phi\rangle \\
&\quad - \frac{(2\alpha + 1)(-1)^k}{2\Delta + 1}\,(S_k)_{ab} L_{-1} G^{\pm b}_{-1/2}|\phi_k\rangle \qquad (4.12) \\
|W\rangle &= \frac{1}{4}\left[\frac{1}{2}\,(G^{+}_{-1/2}G^{+}_{-1/2})(G^{-}_{-1/2}G^{-}_{-1/2}) + 2G^{+}_{-1/2}G^{-}_{-1/2}L_{-1}\right. \\
&\quad \left. - 2(1 - 2\alpha)L_{-1}L_{-1}\right]|\phi\rangle - \frac{1}{4}\,\frac{2\alpha + 1}{\Delta + 1}\,(G^{+}_{-1/2}S_k G^{-}_{-1/2})\phi_k\rangle
\end{aligned}$$

with $\alpha = \alpha^{\pm}(\Delta, \mu, q = 0)$, Eq (4.6a), for $\mu = 0, \ldots, \Delta$.

Furthermore, we define the superstate by

$$\begin{aligned}
|\Phi^q(\theta)\rangle &= |\phi^q\rangle + \theta^{+}|\phi^{q-1}\rangle + \theta^{-}|\phi^{q+1}\rangle - \frac{1}{2\sqrt{2}}\,(\theta^{+})^2|H^{q-2}\rangle \\
&\quad + \frac{1}{2\sqrt{2}}\,(\theta^{-})^2|H^{q+2}\rangle - \frac{1}{2}\,(\theta^{+}\theta^{-})|H^q\rangle \\
&\quad - \frac{(-1)^k}{2}\,(\theta^{+}S_k\theta^{-})|R^q_k\rangle + (\theta^{+}\theta^{-})(\theta^{+}|R^{q-1}\rangle + \theta^{-}|R^{q+1}\rangle)
\end{aligned}$$

$$+ \frac{1}{2} (\theta^{+})^2(\theta^{-})^2|w^q\rangle$$

with

$$|\Phi^q(\theta)\rangle = \Phi^q(0, \theta)|0\rangle \tag{4.13a}$$

The condition for $|\phi^q\rangle$ to be $N = 4$ super-primary can be equivalently stated as that on $|\Phi^q(\theta)\rangle$,

$$\begin{aligned} &\tilde{L}_0|\Phi^q(\theta)\rangle = \Delta|\Phi^q(\theta)\rangle, \qquad \tilde{U}_{k,0}|\Phi^q(\theta)\rangle = -2(-1)^k|\Phi^q_k(\theta)\rangle \\ &\tilde{L}_n|\Phi^q(\theta)\rangle = 0, \qquad \tilde{U}_{k,n}|\Phi^q(\theta)\rangle = 0, \qquad \tilde{G}^{\pm}_{a,r}|\Phi^q(\theta)\rangle = 0 \qquad (n, r > 0) \end{aligned} \tag{4.13b}$$

$$\begin{aligned} &\tilde{U}_0|\Phi^q(\theta)\rangle = q|\Phi^q(\theta)\rangle, \qquad \tilde{U}^{\pm\pm}_0|\Phi^q(\theta)\rangle = 0 \\ &\tilde{U}_n|\Phi^q(\theta)\rangle = 0, \qquad \tilde{U}^{\pm\pm}_n|\Phi^q(\theta)\rangle = 0, \qquad \tilde{j}^{\pm}_{a,r}|\Phi^q(\theta)\rangle = 0 \\ &\tilde{j}^{\pm}_{a,r}|\Phi^q(\theta)\rangle = 0, \qquad \tilde{j}_n|\Phi^q(\theta)\rangle = 0 \qquad (n, r > 0) \end{aligned} \tag{4.13c}$$

Equations (3.10) and (4.11c) or (4.13c) imply that the superfields $\Phi^q_{\pm\nu}$ ($\mu \neq \Delta$) are null superfields for $q \neq 0$. In fact we have

$$\begin{aligned} U^{\pm\pm}_0|\phi^q\rangle = 0 &\Rightarrow \langle\phi^q|U^{--}_0\, U^{++}_0|\phi^q\rangle \\ &= \langle\phi^q|[U^{--}_0, U^{++}_0]|\phi^q\rangle \\ &= -2\langle\phi^q|U_0|\phi^q\rangle = -2q\langle\phi^q|\phi^q\rangle = 0 \end{aligned} \tag{4.14a}$$

Now by using Eqs. (3.10) and (4.12), we find in the case of $\mu = \Delta$ ($\alpha = 1/2$) the following result:

$$\langle W|W\rangle = \langle F^{\pm}_a|F^{\pm}_a\rangle = \langle H|H\rangle = \langle H^{\pm\pm}|H^{\pm\pm}\rangle = 0 \tag{4.14b}$$

We have also for $\nu = \Delta$

$$\langle E_k|E_k\rangle = 0 \qquad \text{for} \quad k = 1$$

for $\Delta = 1$

$$\begin{aligned} &\langle E_{\pm\nu,k}|\, E_{\pm\nu,k}\rangle = 0 \qquad \text{for} \quad k = 2, 3 \text{ and } \nu = 0 \\ &\langle E_{+\nu,k}|\, E_{+\nu,k}\rangle = 0 \qquad \text{for} \quad k = 1, 3 \text{ and } \nu = 1 \\ &\langle E_{-\nu,k}|\, E_{-\nu,k}\rangle = 0 \qquad \text{for} \quad k = 1, 2 \text{ and } \nu = 1 \end{aligned} \tag{4.14c}$$

and for $\Delta = 2$

$$\langle E_{+1,3}|\, E_{+1,3}\rangle = \langle E_{-1,2}|\, E_{-1,2}\rangle = 0$$

Finally, we have the following results:

(i) For $\mu \neq \Delta(\alpha = \alpha^{\pm} \neq (1/2, -1/2))$, the superfields $\Phi^{q}_{\pm\nu}$ are null superfields for $q \neq 0$. The $N = 4$ super-W algebra contains in addition to the generators of the large $N = 4$ superconformal algebra (3.9) the primary fields W, $F^{\pm}_{a}$, E_k, H [Eq. (4.5a), $q = 0$)], $H^{\pm\pm}$, $\phi^{\pm}_{a}$, and ϕ.

(ii) For $\mu = \Delta(\alpha = 1/2)$, the primary fields W, $F^{\pm}$, H, and $H^{\pm\pm}$ are null fields. The primary field E_k is also a null field for certain values of Δ, ν, and k [Eq. (4.14c)]. Notice that this result remains true for $q \neq 0$. So in this case the $N = 4$ super–W algebra contains in addition to the generators of the small $N = 4$ superconformal algebra (L_n, $G^{\pm}_{a,r}$, and $U_{k,n}$) the primary fields E^{q}_{k}, $\phi^{q\pm1}_{a}$, and ϕ^{q}.

5. SOPEs FOR $N = 4$ SUPER-W ALGEBRA

In this section we will give the SOPE of the $N = 4$ super-W algebra in two cases. For the first type ($\mu \neq \Delta$), the algorithm for constructing the $N = 4$ super-W algebra is given in the case of $\mu = 0$. The descendant fields are determined up to level $n = 2$ for the superconformal family of the identity operator and up to level $n = 1$ for the superconformal family of the superfield Φ. So we have an explicit form of the SOPE $\Phi\Phi$ for $\Delta = 1/2$. For the second type ($\mu \mp \Delta$, $\alpha = 1/2$) we give the SOPE for $\Delta = 1$. For the general case ($\mu = 0, \ldots, \Delta$) of the superfields $\Phi = \Phi_{\pm\nu}$ ($\nu = 0, \ldots, \mu$ for μ integer and $\nu = 1/2, \ldots, \mu$ for μ half integer) we give in the Appendix the relations necessary to determine the algorithm and consequently the coefficients of the descendant fields. The SOPEs we will consider are

$$\Phi\Phi, \quad \mu = 0; \qquad \Phi_i\Phi_j \quad (i, j = 1, 2, 3), \quad \mu = 1 \tag{5.1}$$

with $\Phi_0 = \Phi$ for $\mu = 0$ and $\Phi_0 = \Phi_1$, $\Phi_{+1} = \Phi_2$, and $\Phi_{-1} = \Phi_3$ for $\mu = 1$.

In the case of $\mu = 1$ the SOPE take the following form

$$\begin{aligned}
&\Phi_i(Z_1)\Phi_j(Z_2) \\
&\quad = \sum_{n=0}^{\infty} \frac{1}{Z_{12}^{\Delta_m - n}} \Bigg[\Phi^{n}_{m,n,(i,j)}(Z_2) + \theta^{+}_{12}\, \Phi^{-(n+1/2)}_{m,n,\,(i,j)}(Z_2) + \theta^{-}_{12}\Phi^{+(n+1/2)}_{m,n,(i,j)}(Z_2) \\
&\qquad + \frac{(\theta^{+}_{12})^2}{Z_{12}}\, \Phi^{--n}_{m,n-\ 1,(i,j)}(Z_2) + \frac{(\theta^{-}_{12})^2}{Z_{12}}\, \Phi^{++n}_{m,n-1,(i,\,j)}(Z_2) \\
&\qquad + \frac{(\theta^{+}_{12}\theta^{-}_{12})}{Z_{12}}\, \Phi^{n}_{m,n-1(i,j)}(Z_2) + \frac{(\theta^{+}_{12}S_{(i}\theta^{-}_{12})}{Z_{12}}\, \Phi^{n}_{m,n-1(i,j)}(Z_2) \\
&\qquad + \frac{(\theta^{+}_{12}S_k\theta^{-}_{12})}{Z_{12}}\, \Phi^{n}_{m,n-1(i,j),k}(Z_2) + \frac{(\theta^{+}_{12}\theta^{-}_{12})}{Z_{12}}\, \theta^{+}_{12}\Phi^{-(n+1/2)}_{m,n-1,(i,j)}(Z_2)
\end{aligned}$$

$$+ \frac{(\theta^+_{12}\theta^-_{12})}{Z_{12}}\,\theta^-_{12}\Phi^{+(n+1/2)}_{m,n-1,(i,j)}(Z_2) + \frac{(\theta^+_{12})^2(\theta^-_{12})^2}{Z^2_{12}}\,\Phi^n_{m,n-2,(i,j)}(Z_2)\Bigg] \tag{5.2a}$$

with

$$\begin{gathered} m = 1, 2; \qquad \Delta_1 = 2\Delta, \qquad \Delta_2 = \Delta; \\ \Phi^0_{1,0,(i,j)} = I^0_{1,0,(i,j)}, \qquad \Phi^0_{2,0,(i,j)} = \Phi^0_{0,(i,j)} \end{gathered} \tag{5.2b}$$

which are the descendant fields of the identity operator and the superfields Φ_i, Φ_j, and Φ_k with $k \neq i, j$ and $i, j, k = 1, 2, 3$. The expression (5.2a) is valid for the two types of $N = 4$ super-W algebra ($\mu \neq \Delta$ and $\mu = \Delta$). So by using the recursion relations (A.2a) we have determined the SOPE (5.2a) for $\Delta = \mu = 1$, which is given by

$$\begin{aligned}
&\Phi_i(Z_1)\Phi_j(Z_2) \\
&\quad = \frac{c}{3}\frac{(-1)^i\delta_{ij}}{Z^2_{12}} - \frac{2c}{3}(-1)^k\,\varepsilon^{ijk}\,\frac{(\theta^+_{12}S_k\theta^-_{12})}{Z_{12}} + \frac{c}{2}\frac{(\theta^+_{12})^2(\theta^-_{12})^2}{Z^4_{12}} \\
&\qquad + \frac{\theta^{+a}_{12}}{Z_{12}}[(-1)^i\delta_{ij}\tilde{G}^-_{a,-3/2}\,I(Z_2) + \varepsilon^{ijk}(S_k)_{ab}\,\tilde{G}^{-b}_{-3/2}I(Z_2) - (S_i)_{ab}\,\tilde{G}^{-b}_{-3/2}\Phi_j(Z_2)] \\
&\qquad - \frac{\theta^{-a}_{12}}{Z_{12}}[(-1)^i\delta_{ij}\tilde{G}^+_{a,-3/2}\,I(Z_2) + \varepsilon^{ijk}(S_k)_{ab}\tilde{G}^{+b}_{-3/2}I(Z_2) - (S_i)_{ab}\tilde{G}^{+b}_{-3/2}\,\Phi_j(Z_2)] \\
&\qquad + 2(-1)^k\frac{\varepsilon^{ijk}}{Z_{12}}[\tilde{U}_{k,-1}I(Z_2) + \Phi_k(Z_2)] \\
&\qquad - 2\,\frac{(\theta^+_{12}S_i\theta^-_{12})}{Z_{12}}[\tilde{U}_{j,-2}I(Z_2) + \tilde{L}_{-1}\Phi_j(Z_2)] \\
&\qquad + 2(-1)^{i+k}\delta_{ij}\,\frac{(\theta^+_{12}S_k\theta^-_{12})}{Z^2_{12}}[\tilde{U}_{k,-1}I(Z_2) + \Phi_k(Z_2)] \\
&\qquad - 2\,\frac{(\theta^+_{12}S_{(i}\theta^-_{12})}{Z^2_{12}}[\tilde{U}_{j),-1}I(Z_2) \\
&\qquad + \Phi_{j)}(Z_2)] + \frac{(\theta^+_{12}\theta^-_{12})}{Z^2_{12}}[\theta^{+a}_{12}((-1)^i\delta_{ij}\tilde{G}^-_{a,-3/2}I(Z_2) + \varepsilon^{ijk}(S_k)_{ab}\tilde{G}^{-b}_{-3/2}I(Z_2)] \\
&\qquad - (S_i)_{ab}\,\tilde{G}^{-b}_{-3/2}\Phi_j(Z_2)) + \theta^{-a}_{12}((-1)^i\delta_{ij}\tilde{G}^+_{a,-3/2}I(Z_2) + \varepsilon^{ijk}(S_k)_{ab}\tilde{G}^{+b}_{-3/2}I(Z_2) \\
&\qquad - (S_i)_{ab}\tilde{G}^{+b}_{-3/2}\Phi_j(Z_2))] + (-1)^k\varepsilon^{ijk}\,\frac{(\theta^+_{12})^2(\theta^-_{12})^2}{Z^3_{12}}[\tilde{U}_{k,-1}I(Z_2) + \Phi_k(Z_2)]
\end{aligned} \tag{5.3}$$

where $\tilde{L}_n$, $\tilde{G}^{\pm}_{a,r}$, and $\tilde{U}_{k,n}$ are defined by (4.10b). Now we focus on the first type of $N = 4$ super-W algebra ($\mu \neq \Delta$) for $\mu = 0$. In this case we have $\alpha = \alpha^{\pm} = 0$ and $c' = 0$. This means that the central extension of the $N = 4$ superconformal algebra is reduced to one ($C_0 = c$). The SOPE in this case take the following form:

$$\Phi(Z_1)\Phi(Z_2)$$
$$= \sum_{n=0}^{\infty} \frac{1}{Z_{12}^{\Delta_m - n}} \Bigg[\Phi^n_{m,n}(Z_2) + \theta^+_{12}\Phi^{-(n+1/2)}_{m,n}(Z_2) + \theta^-_{12}\Phi^{+(n+1/2)}_{m,n}(Z_2)$$
$$+ \frac{(\theta^+_{12})^2}{Z_{12}} \Phi^{--n}_{m,n-1}(Z_2) + \frac{(\theta^-_{12})^2}{Z_{12}} \Phi^{++n}_{m,\,n-1}(Z_2) + \frac{(\theta^+_{12}\theta^-_{12})}{Z_{12}} \Phi^n_{m,n-1}(Z_2)$$
$$+ \frac{(\theta^+_{12} S_i \theta^-_{12})}{Z_{12}} \Phi^n_{m,n-1,i}(Z_2) + \frac{(\theta^+_{12}\theta^-_{12})}{Z_{12}} (\theta^+_{12}\Phi^{-(n+1/2)}_{m,n-1}(Z_2)$$
$$+ \theta^-_{12}\Phi^{+(n+1/2)}_{m,n-1}(Z_2)) + \frac{(\theta^+_{12})^2(\theta^-_{12})^2}{Z^2_{12}} \Phi^n_{m,n-2}(Z_2) \Bigg] \tag{5.4}$$

The algorithm in this case is given by

$$\tilde{L}_n\Phi^p_{m,p} = (\Delta(n+1) - \Delta_m + p - n)\Phi^{p-n}_{m,p-n}$$
$$\tilde{L}_n\Phi^{\pm(p+1/2)}_{m,\,p} = \left(\Delta(n+1) - \Delta_m + p - \frac{1}{2}(n-1)\right)\Phi^{\pm(p-n+1/2)}_{m,\,p-n}$$
$$\tilde{L}_n\Phi^{\pm\pm p}_{m,p-1} = (\Delta(n+1) - \Delta_m + p)\Phi^{\pm\pm(p-n)}_{m,\,p-n-1}$$
$$\tilde{L}_n\Phi^p_{m,p-1} = (\Delta\,(n+1) - \Delta_m + p)\Phi^{p-n}_{m,p-n-1}$$
$$\tilde{L}_n\Phi^p_{m,p-1,k} = (\Delta\,(n+1) - \Delta_m + p)\Phi^{p-n}_{m,p-n-1,k}$$
$$\tilde{L}_n\Phi^{\pm(p+1/2)}_{m,p\,-1} = \left(\left(\Delta + \frac{1}{2}\right)(n+1) - \Delta_m + p\right)\Phi^{\pm(p-n+1/2)}_{m\,,p-n-1}$$
$$\tilde{L}_n\Phi^p_{m,p-2} = ((\Delta+1)(n+1) - \Delta_m + p)\Phi^{p-n}_{m,p-n\,-2}$$
$$G^{\pm}_{a,r}\,\Phi^p_{m,\,p} = \pm\Phi^{\pm(p-r)}_{m,p-r-1/2,a}$$
$$G^{\pm}_{a,r}\Phi^{\pm(p+1/2)}_{m,p,b} = \mp 2\,\varepsilon_{ab}\Phi^{\pm\pm(p-r+1/2)}_{m,\,p-r-1/2}$$
$$G^{\pm}_{a,r}\Phi^{\pm\pm p}_{m,p-1} = 0$$
$$G^{\pm}_{a,r}\Phi^{\mp(p+1/2)}_{m,p,b} = \mp\varepsilon_{ab}\Phi^{(p-r+1/2)}_{m,p-r-1/2} + (-1)^k(S_k)_{ab}\,\Phi^{(p-r+1/2)}_{m,p-r-1/2,k}$$
$$+ \varepsilon_{ab}\left(2\Delta\left(r + \frac{1}{2}\right) - \Delta_m + p - r + \frac{1}{2}\right)\Phi^{(p-r+1/2)}_{m,p-r+1/2}$$

$$G^{\pm}_{a,r}\Phi^{\mp\mp p}_{m,p-1} = \mp\frac{1}{2}\,\Phi^{\mp(p-r)}_{m,p-r-3/2,a} - \frac{1}{2}\left(2\Delta\left(r+\frac{1}{2}\right) - \Delta_m + p\right)\Phi^{\mp(p-r)}_{m,p-r-1/2,a}$$

$$G^{\pm}_{a,r}\Phi^{p}_{m,p-1} = \pm\frac{1}{2}\,\Phi^{\pm(p-r)}_{m,p-r-3/2,a} - \frac{1}{2}\left(2\Delta\left(r+\frac{1}{2}\right) - \Delta_m + p\right)\Phi^{\pm(p-r)}_{m,p-r-1/2,a}$$

$$G^{\pm}_{a,r}\Phi^{p}_{m,p-1,k} = \frac{1}{2}\,(S_k)_{ab}\left[\,\Phi^{\pm(p-r)b}_{m,p-r-3/2} \mp \left(2\Delta\left(r+\frac{1}{2}\right) - \Delta_m + p\right)\Delta^{\pm(p-r)b}_{m,p-r-1/2}\right]$$

$$G^{\pm}_{a,r}\Phi^{\pm(p+1/2)}_{m,p-1,b} = 2\varepsilon_{ab}\left((2\Delta+1)\left(r+\frac{1}{2}\right) - \Delta_m + p\right)\Phi^{\pm\pm(p-r+1/2)}_{m,p-r-1/2}$$

$$\begin{aligned} G^{\pm}_{a,r}\Phi^{\mp(p+1/2)}_{m,p-1,b} = {} & \pm 4\varepsilon_{ab}\Phi^{(p-r+1/2)}_{m,p-r-3/2} \\ & - \left((2\Delta+1)\left(r+\frac{1}{2}\right) - \Delta_m + p\right)(\varepsilon_{ab}\Phi^{(p-r+1/2)}_{m,p-r-1/2} \\ & \mp (-1)^k (S_k)_{ab}\Phi^{(p-r+1/2)}_{m,p-r-1/2,k}) \end{aligned}$$

$$G^{\pm}_{a,r}\Phi^{p}_{m,p-2} = \frac{1}{4}\left(2(\Delta+1)\left(r+\frac{1}{2}\right) - \Delta_m + p - 1\right)\Phi^{\pm(p-r)}_{m,p-r-3/2,a}$$

$$\tilde{U}_{k,n}\Phi^{p}_{m,p} = 0$$

$$\tilde{U}_{k,n}\Phi^{\pm(p+1/2)}_{m,p,a} = -(-1)^k(S_k)_{ab}\Phi^{\pm(p-n+1/2)b}_{m,p-n}$$

$$\tilde{U}_{k,n}\Phi^{\pm\pm p}_{m,p-1} = 0$$

$$\tilde{U}_{k,n}\Phi^{p}_{m,p-1} = 0$$

$$\tilde{U}_{k,n}\Phi^{p}_{m,p-1,i} = 2n\Delta\delta_{ki}\Phi^{p-n}_{m,p-n} - 2(-1)^j\varepsilon^{kij}\Phi^{p-n}_{m,p-n-1,j}$$

$$\tilde{U}_{k,n}\Phi^{\pm(p+1/2)}_{m,p-1,a} = -(-1)^k(S_k)_{ab}(\Phi^{\pm(p-n+1/2)b}_{m,p-n-1} \mp (2\Delta+1)n\Phi^{\pm(p-n+1/2)b}_{m,p-n})$$

$$\tilde{U}_{k,n}\Phi^{p}_{m,p-2} = -(-1)^k n(\Delta+1)\Phi^{p-n}_{m,p-n-1,k}$$

$$\tilde{U}_{n}\Phi^{p}_{m,p} = 0$$

$$\tilde{U}_{n}\Phi^{\pm(p+1/2)}_{m,p,a} = \pm\Phi^{\pm(p-n+1/2)b}_{m,p-n}$$

$$\tilde{U}_{n}\Phi^{\pm\pm p}_{m,p-1} = \pm 2\Phi^{\pm\pm(p-n)}_{m,p-n-1}$$

$$\tilde{U}_{n}\Phi^{p}_{m,p-1} = 2n\Delta\Phi^{p-n}_{m,p-n-1}$$

$$\tilde{U}_{n}\Phi^{p}_{m,p-1,i} = 0$$

$$\tilde{U}_{n}\Phi^{\pm(p+1/2)}_{m,p-1,a} = \pm\Phi^{\pm(p-n+1/2)}_{m,p-n-1,a} + (2\Delta+1)n\Phi^{\pm(p-n+1/2)}_{m,p-n,a}$$

$$
\begin{aligned}
\tilde{U}_n \Phi^p_{m,p-2} &= -n(\Delta + 1)\Phi^{p-n}_{m,p-n-1} \\
\tilde{U}^{\pm\pm}_n \Phi^p_{m,p} &= 0 \\
\tilde{U}^{\pm\pm}_n \Phi^{\pm(p+1/2)}_{m,p,a} &= 0 \\
\tilde{U}^{\pm\pm}_n \Phi^{\mp(p+1/2)}_{m,p,a} &= \sqrt{2}\, \Phi^{\pm(p-n+1/2)}_{m,p-n,a} \\
\tilde{U}^{\pm\pm}_n \Phi^{\mp\mp p}_{m,p-1} &= \sqrt{2}\, \Phi^{p-n}_{m,p-n-1} \pm n\Delta\Phi^{p-n}_{m,p-n} \\
\tilde{U}^{\pm\pm}_n \Phi^{\pm\pm p}_{m,p-1} &= 0 \\
\tilde{U}^{\pm\pm}_n \Phi^p_{m,p-1} &= 2\sqrt{2}\, \Phi^{\pm\pm p-n}_{m,p-n-1} \\
\tilde{U}^{\pm\pm}_n \Phi^p_{m,p-1,i} &= 0 \\
\tilde{U}^{\pm\pm}_n \Phi^{\pm(p+1/2)}_{m,p-1,a} &= 0 \\
\tilde{U}^{\pm\pm}_n \Phi^{\mp(p+1/2)}_{m,p-1,a} &= -2\sqrt{2}\, \Phi^{\pm(p-n+1/2)}_{m,p-n-1,a} \mp n\sqrt{2}\,(2\Delta + 1)n\Phi^{\pm(p-n+1/2)}_{m,p-n,a} \\
\tilde{U}^{\pm\pm}_n \Phi^p_{m,p-2} &= \pm n\sqrt{2}\,(\Delta + 1)\Phi^{\pm\pm p-n}_{m,p-n-1} \\
j^{\pm}_{a,r}\Phi^p_{m,p} &= \pm\Phi^{\pm(p-r)}_{m,p-r-1/2,a} \\
j^{\pm}_{a,r}\Phi^{\pm(p+1/2)}_{m,p,b} &= 0 \\
j^{\pm}_{a,r}\Phi^{\mp(p+1/2)}_{m,p,b} &= 0 \\
j^{\pm}_{a,r}\Phi^{\mp\mp p}_{m,p-1} &= \pm\frac{1}{2}\,\Phi^{\mp(p-r)}_{m,p-r-1/2,a} \\
j^{\pm}_{a,r}\Phi^{\pm\pm p}_{m,p-1} &= 0 \\
j^{\pm}_{a,r}\Phi^p_{m,p-1} &= \pm\frac{1}{2}\,\Phi^{\pm(p-r)}_{m,p-r-1/2,a} \\
j^{\pm}_{a,r}\Phi^p_{m,p,-1,k} &= -\frac{1}{2}\,(S_k)_{ab}\Phi^{\pm(p-r)b}_{m,p-r-1/2} \\
j^{\pm}_{a,r}\Phi^{\pm(p+1/2)}_{m,p-1,b} &= \pm 2\varepsilon_{ab}\Phi^{\pm\pm(p-r+1/2)}_{m,p-r-1/2} \\
j^{\pm}_{a,r}\Phi^{\mp(p+1/2)}_{m,p-1,b} &= \mp\varepsilon_{ab}\Phi^{(p-r+1/2)}_{m,p-r-1/2} - (-1)^k(S_k)_{ab}\Phi^{(p-r+1/2)}_{m,p-r-1/2,k} \\
&\quad + \left((2\Delta + 1)\left(r - \frac{1}{2}\right) - \Delta_m - p\right)\varepsilon_{ab}\Phi^{(p-r+1/2)}_{m,p-r+1/2} \\
j^{\pm}_{a,r}\Phi^p_{m,p-2} &= -\frac{1}{4}\left(2(\Delta + 1)\left(r - \frac{1}{2}\right) + \Delta_m - p + 1\right)\Phi^{\pm(p-r)}_{m,p-r-1/2,a}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{j}_n \Phi^{p}_{m,p} &= 0 \\
\tilde{j}_n \Phi^{\pm(p+1/2)}_{m,p,a} &= 0 \\
\tilde{j}_n \Phi^{\pm\pm p}_{m,p-1} &= 0 \\
\tilde{j}_n \Phi^{p}_{m,p-1} &= 0 \\
\tilde{j}_n \ \Phi^{p}_{m,p-1,i} &= 0 \\
\tilde{j}_n \Phi^{\pm(p+1/2)}_{m,p-1,a} &= \mp \Phi^{\pm(p-n+1/2)}_{m,p-n,a} \\
\tilde{j}_n \Phi^{p}_{m,p-2} &= \frac{1}{2}\left(\Delta(n-1) + \Delta_m - p + n\right)\Phi^{p-n}_{m,p-n}
\end{aligned} \tag{5.5}
$$

with $m = 1, 2$; $\Delta_1 = 2\Delta$, $\Delta_2 = \Delta$, $\Phi^{0}_{1,()} = I^{0}_{1,()}$, and $\Phi^{0}_{2,()} = \Phi^{0}_{()}$. Notice that for Δ half-integer we have simply the superconformal family of the identity operator. The descendants fields of the identity operator up to level $n = 2$ are

$$
\begin{aligned}
I^{0}_{0} &= \frac{c}{3\Delta}, \qquad I^{\pm 1/2}_{0,\mathrm{a}} = 0, \\
I^{\pm\pm 0}_{-1} &= I^{0}_{-1} = I^{0}_{-1,\mathrm{i}} = 0 = J^{0}_{-2}, \\
I^{\pm 1/2}_{-1,a} &= \pm 2\tilde{j}^{\pm}_{a,-1/2}
\end{aligned} \tag{5.6a}
$$

for $n = 0$

$$
I^{1}_{1}(Z) = 0, \qquad I^{\pm\pm 0}_{0}(Z) = \frac{c}{\sqrt{2}(c-3)}\left(\mp \tilde{U}^{\pm\pm}_{-1}(Z) - \frac{3}{\sqrt{2}c}\tilde{j}^{\pm}_{-1/2}\tilde{j}^{\pm}_{-1/2}(Z)\right)
$$

$$
I^{\pm\pm 1}_{0}(Z) = \frac{c}{\sqrt{2}(c-3)}\left[\mp \tilde{U}^{\pm\pm}_{-1} - \frac{3}{\sqrt{2}c}\tilde{j}^{\pm}_{-1/2}\tilde{j}^{\pm}_{-1/2}\right](Z)
$$

$$
I^{1}_{0}(Z) = \frac{c}{c-3}\left[\tilde{U}_{-1} - \frac{3}{c}\tilde{j}^{+}_{-1/2}\tilde{j}^{-}_{-1/2}\right](Z)
$$

$$
I^{1}_{0,k}(Z) = -\frac{(-1)^k c}{c-3}\left[\tilde{U}_{k,-1} - \frac{3}{c}\tilde{j}^{+}_{-1/2}S_k\tilde{j}^{-}_{-1/2}\right](Z)
$$

$$
I^{1}_{-1}(Z) = \tilde{j}_{-1}(Z), \qquad I^{1}_{1}(Z) = 0
$$

$$
\begin{aligned}
I^{\pm 3/2}_{1,a}(Z) = \frac{c}{(c-3)^2}\Bigg[& \pm \tilde{G}^{\pm}_{a,-3/2} \mp \frac{3}{2}\tilde{U}_{-1}\tilde{j}^{\pm}_{a,-1/2} + 3\tilde{j}_{-1}\tilde{j}^{\pm}_{a,-1/2} \\
& - \frac{3}{\sqrt{2}}\tilde{U}^{\pm\pm}_{-1}\tilde{j}^{\mp}_{a,-1/2} + \frac{3}{2}(S_k)_{ab}\tilde{U}_{k,-1}\tilde{j}^{\pm b}_{-1/2}
\end{aligned}
$$

$$\pm \frac{18}{c}\tilde{j}^{\pm}_{-1/2}\tilde{j}^{-}_{-1/2}\, j^{\pm}_{a,-1/2}\Big](Z)$$

$$I^{\pm 3/2}_{0,a}(Z) = \frac{1}{c-3}\Big[\pm 2(2c+3)\tilde{j}^{\pm}_{a,-3/2} - 3\tilde{U}_{-1}\tilde{j}^{\pm}_{a,-1/2} + 3\tilde{j}_{-1}\tilde{j}^{\pm}_{a,-1/2}$$

$$\mp 3\sqrt{2}\tilde{U}^{\pm\pm}_{-1}\tilde{j}^{\mp}_{a,-1/2} \mp 3(S_k)_{ab}\,\tilde{U}_{k,-1}\,\tilde{j}^{\pm b}_{-1/2}\Big](Z) \qquad (5.6b)$$

for $n = 1$ and

$$I^{2}_{2}(Z) = \frac{c}{(c-3)^2}\Big[\frac{2(c+3)}{3c}\tilde{L}_{-2} + \tilde{U}_{-2} - \frac{1}{2}\tilde{U}_{-1}\tilde{U}_{-1} - \frac{c+3}{c}\tilde{j}_{-1}\tilde{j}_{-1}$$

$$+ \frac{(-1)^k}{2}\tilde{U}_{k,-1}\tilde{U}_{k,-1} - \tilde{U}^{++}_{-1}\tilde{U}^{--}_{-1} + \frac{3}{c}\tilde{U}_{-1}\tilde{j}^{+}_{-1/2}\tilde{j}^{-}_{-1/2}$$

$$- \frac{3(-1)^k}{c}\tilde{U}_{k,-1}\tilde{j}^{+}_{-1/2}S_k\tilde{j}^{-}_{-1/2} + \frac{3}{\sqrt{2}\,c}\tilde{U}^{++}_{-1}\tilde{j}^{-}_{-1/2}\tilde{j}^{-}_{-1/2}$$

$$- \frac{3}{\sqrt{2}c}\tilde{U}^{--}_{-1}\tilde{j}^{+}_{-1/2}\tilde{j}^{+}_{-1/2} - \frac{c+12}{c}\tilde{j}^{+}_{-3/2}\tilde{j}^{-}_{-1/2} + \frac{c+12}{c}\tilde{j}^{-}_{-3/2}\tilde{j}^{+}_{-1/2}\Big](Z)$$

$$I^{2}_{1}(Z) = \frac{c}{(c-3)^2}\Big[c\tilde{U}_{-2} - \frac{3}{2}\tilde{G}^{+}_{-3/2}\tilde{j}^{-}_{-1/2} + \frac{3}{2}\tilde{G}^{-}_{-3/2}\tilde{j}^{+}_{-1/2} + \frac{9}{c}\tilde{j}_{-1}\tilde{j}^{+}_{-1/2}\tilde{j}^{-}_{-1/2}$$

$$- \frac{9}{2\sqrt{2c}}\tilde{U}^{++}_{-1}\tilde{j}^{-}_{1/2}\tilde{j}^{-}_{1/2} + \frac{9}{2\sqrt{2c}}\tilde{U}^{--}_{-1}\tilde{j}^{+}_{-1/2}\tilde{j}^{+}_{-1/2}$$

$$- 3\tilde{j}^{+}_{-3/2}\tilde{j}^{-}_{-1/2} - 3\tilde{j}^{-}_{-3/2}\tilde{j}^{+}_{-1/2}\Big](Z)$$

$$I^{2}_{0}(Z) = \frac{3c}{(c+3)(c-3)}\Big[\frac{\Delta+1}{2}\tilde{U}_{-2} + \frac{(c+3)(c^2-4(\Delta+1)c+9)}{c(c-3)}\tilde{j}_{-2}$$

$$- \frac{\Delta+1}{4}\tilde{U}_{-1}\tilde{U}_{-1} - \frac{(-1)^k(\Delta+1)}{2}\tilde{U}_{k,-1}\tilde{U}_{k,-1} - \frac{\Delta+1}{2}\tilde{U}^{++}_{-1}\tilde{U}^{--}_{-1}$$

$$+ \frac{3((4\Delta+1)c-9)}{4c(c-3)}\tilde{U}_{-1}\tilde{j}^{+}_{-1/2}\tilde{j}^{-}_{-1/2}$$

$$+ \frac{3((4\Delta+1)c-9)}{4c(c-3)}(-1)^k\tilde{U}_{k,-1}\tilde{j}^{+}_{-1/2}S_k\tilde{j}^{-}_{-1/2}$$

$$+ \frac{3((4\Delta+1)c-9)}{4\sqrt{2}c(c-3)} \tilde{U}^{++}_{-1} \tilde{j}^{-}_{-1/2} \tilde{j}^{-}_{-1/2}$$

$$- \frac{3((4\Delta+1)c-9)}{4\sqrt{2}c(c-3)} \tilde{U}^{--}_{-1} \tilde{j}^{+}_{-1/2} \tilde{j}^{+}_{-1/2}$$

$$- \frac{3((2\Delta-1)(c+3))}{4c(c-3)} \tilde{G}^{+}_{-3/2} \tilde{j}^{-}_{-1/2}$$

$$- \frac{3((2\Delta-1)(c+3))}{4c(c-3)} \tilde{G}^{-}_{-3/2} \tilde{j}^{+}_{-1/2} \Bigg](Z)$$

$$I^{2}_{1,k}(Z) = -\frac{3}{(c-3)^2} \Bigg[\lambda \tilde{U}_{k,-2} + \eta \varepsilon^{kij} \tilde{U}_{i,-1} \tilde{U}_{j,-1}$$

$$+ \frac{3}{2} \varepsilon^{kij} \tilde{U}_{i,-1} \tilde{j}^{+}_{-1/2} S_j \tilde{j}^{-}_{-1/2} - 3(-1)^k \tilde{j}_{-1} \tilde{j}^{+}_{-1/2} S_k \tilde{j}^{-}_{-1/2}$$

$$+ (-1)^k c \tilde{j}^{+}_{-3/2} S_k \tilde{j}^{-}_{-1/2} - (-1)^k c \tilde{j}^{-}_{-3/2} S_k \tilde{j}^{+}_{-1/2}$$

$$- \frac{(-1)^k c}{2} \tilde{G}^{+}_{-3/2} S_k \tilde{j}^{-}_{-1/2} - \frac{(-1)^k c}{2} \tilde{G}^{-}_{-3/2} S_k \tilde{J}^{+}_{-1/2} \Bigg](Z)$$

with $\lambda - 2\eta = -c^2/(c-3)^2$ and

$$I^{\pm\pm 2}_{0}(Z) = \frac{\mathrm{c}}{(c-3)^2} \Bigg[\mp \frac{c}{\sqrt{2}} \tilde{U}^{\pm\pm}_{-1} - \frac{9}{4c} \tilde{U}_{-1} \tilde{j}^{\pm}_{-1/2} \tilde{j}^{\pm}_{-1/2}$$

$$+ \frac{9}{2\sqrt{2c}} \tilde{U}^{\pm\pm}_{-1} \tilde{j}^{+}_{-1/2} \tilde{j}^{-}_{-1/2}$$

$$- \frac{9}{2\mathrm{c}} \tilde{j}_{-1} \tilde{j}^{\pm}_{-1/2} \tilde{j}^{\pm}_{-1/2} - 3 \tilde{j}^{\pm}_{-3/2} \tilde{j}^{\pm}_{-1/2} \mp \frac{3}{2} \tilde{G}^{\pm}_{-3/2} \tilde{j}^{\pm}_{-1/2} \Bigg](Z) \qquad (5.6c)$$

for $n = 2$.

So the relations (5.6a)–(5.6c) give an explicit form of the SOPE $\Phi\Phi$ for $\Delta = 1/2$. The descendant fields of the superconformal family of superfields up to level $n = 1$ are

$$\Phi^{0}_{0}(Z) = f\Phi(Z), \qquad \Phi^{0}_{-1}(Z) = g\Phi(Z), \qquad \Phi^{0}_{-2} = h\Phi(Z)$$

$$\Phi^{\pm\pm 0}_{-1}(Z) = 0 = \Phi^{0}_{-1,i}(Z), \qquad \Phi^{\pm 1/2}_{0,a} = \frac{\pm\Delta f - g}{2\Delta} \tilde{G}^{\pm}_{a,-1/2}\Phi(Z) \qquad (5.7a)$$

$$\Phi^{\pm 1/2}_{-1,a} = \left[3\,\frac{\pm\Delta f - g}{c}\,\tilde{J}^{\pm}_{a,-1/2} + \frac{\pm(\Delta + 1)g + 4h}{2\Delta}\,\tilde{G}^{\pm}_{a,-1/2}\right]\Phi(Z)$$

for $n = 0$ and

$$\begin{aligned}
\Phi^1_1(Z) = {} & [\beta_1^{\{1,0,0\}}\tilde{L}_{-1} + \beta_1^{\{0,1,0\}}\tilde{U}_{-1} + \beta_1^{\{0,0,1\}}\tilde{J}_{-1} \\
& + \beta_1^{\{0,+1/2,-1/2\}}\tilde{G}^{+}_{-1/2}\tilde{G}^{-}_{-1/2} + \beta_1^{\{0,+1/2,0,-1/2\}}\tilde{G}^{+}_{-1/2}\tilde{J}^{-}_{-1/2} \\
& + \beta_1^{\{0,-1/2,0,+1/2\}}\tilde{G}^{-}_{-1/2}\tilde{J}^{+}_{-1/2} + \beta_1^{\{0,0,+1/2,-1/2\}}\tilde{J}^{+}_{-1/2}\tilde{J}^{-}_{-1/2}]\Phi(Z)
\end{aligned}$$

$$\begin{aligned}
\beta_1^{\{1,0,0\}} = {} & \frac{\Delta + 2}{4\Delta} \\
& \times \frac{((\Delta + 1)g + 2(2\Delta + 1)h)c^2 - 3((3\Delta + 1)g + 2(4\Delta + +3)h)c + 36h}{(2\Delta + 1)c^2 - 3(4\Delta + 3)c + 18}
\end{aligned}$$

$$\beta_1^{\{0,1,0\}} = \frac{\Delta + 2}{4\Delta}\,\frac{(-\Delta^2 + \Delta + 1)c - 3(\Delta + 1)}{(2\Delta + 1)c^2 - 3(4\Delta - 3)c + 18}, \qquad \beta_1^{\{0,0,1\}} = 0$$

$$\beta_1^{\{0,+1/2,-1/2\}} = -\frac{(\Delta + 2)g}{8\Delta}\,\frac{(\Delta + 1)c^2 - 3(\Delta + 1)c}{(2\Delta + 1)c^2 - 3(4\Delta + 3)c + 18}$$

$$\beta_1^{\{0,\pm 1/2,0,\mp 1/2\}} = \mp\frac{3(\Delta + 2)g}{8\Delta}\,\frac{(\Delta + 1)c - 3(3\Delta + 1)}{(2\Delta + 1)c^2 - 3(4\Delta + 3)c + 18}$$

$$\beta_1^{\{0,0,+1/2,-1/2\}} = \mp\frac{9(\Delta + 2)g}{4c}\,\frac{\Delta c - 6}{(2\Delta + 1)c^2 - 3(4\Delta + 3)c + 18}$$

$$\begin{aligned}
\Phi^1_0(Z) = {} & [\beta_0^{\{1,0,0\}}\tilde{L}_{-1} + \beta_0^{\{0,1,0\}}\tilde{U}_{-1} + \beta_0^{\{0,0,1\}}\tilde{J}_{-1} \\
& + \beta_0^{\{0,+1/2,-1/2\}}\tilde{G}^{+}_{-1/2}\tilde{G}^{-}_{-1/2} + \beta_0^{\{0,+1/2,0,-1/2\}}\tilde{G}^{+}_{-1/2}\tilde{J}^{-}_{-1/2} \\
& + \beta_0^{\{0,-1/2,0,+1/2\}}\tilde{G}^{-}_{-1/2}\tilde{J}^{+}_{-1/2} + \beta_0^{\{0,0,+1/2,-1/2\}}\tilde{J}^{+}_{-1/2}\tilde{J}^{-}_{-1/2}]\Phi(Z)
\end{aligned}$$

$$\begin{aligned}
\beta_0^{\{1,0,0\}} = {} & \frac{1}{2\Delta((2\Delta + 1)c^2 - 3(4\Delta + 3)c + 18)} \\
& \times [((2\Delta + 1)(\Delta + 1)g - \Delta(\Delta + 1)f + 4h)c^2 \\
& + 3((5\Delta + 3)\Delta f + (4\Delta + 3)(\Delta + 1)g - 4\lambda)c \\
& + 18(\Delta + 1)g - \Delta f]
\end{aligned}$$

$$\beta_0^{\{0,1,0\}} = \frac{3}{2\Delta}\,\frac{((3\Delta + 2)\Delta^2 f + 4(\Delta + 1)h)\,c - 3(\Delta^2 f + 4h)}{(2\Delta + 1)c^2 - 3(4\Delta + 3)c + 18},$$

$$\beta_1^{\{0,0,1\}} = \frac{3g}{\Delta c}$$

$$\beta_0^{\{0,+1/2,-1/2\}} = \frac{1}{4\Delta}\frac{((\Delta+1)\Delta f - 4h)c^2 - 3((5\Delta+3)\Delta f - 4h)c + 18\Delta f}{(2\Delta+1)c^2 - 3(4\Delta+3)\,c + 18}$$

$$\beta_0^{\{0,\pm 1/2,0,\mp 1/2\}} = \frac{3}{4\Delta c} \times \frac{((2\Delta+1)g \mp \Delta^2 f \mp 4h)c^2 - 3((4\Delta+3)g \pm \Delta^2 g \mp 4h)c + 18g}{(2\Delta+1)c^2 - 3(4\Delta+3)c + 18}$$

$$\beta_0^{\{0,0,+1/2,-1/2\}} = -\frac{9}{2c}\frac{((3\Delta+1)\Delta f + 4h)c - 6\Delta f}{(2\Delta+1)c^2 - 3(4\Delta+3)c + 18}$$

$$\Phi^1_{0,k}(Z) = [\beta_{0,k}^{\{0,1,0\}}\tilde{U}_{k,-1} + \beta_{0,k}^{\{0,+1/2,-1/2\}}\tilde{G}^+_{-1/2}S_k\tilde{G}^-_{-1/2} + \beta_{0,k}^{\{0,+1/2,0,-1/2\}}\tilde{G}^+_{-1/2}S_k\tilde{j}^-_{-1/2} + \beta_{0,k}^{\{0,-1/2,0,+1/2\}}\tilde{G}^-_{-1/2}S_k\tilde{j}^+_{-1/2} + \beta_{0,k}^{\{0,0,+1/2,-1/2\}}\tilde{j}^+_{-1/2}S_k\tilde{j}^-_{-1/2}]\Phi(Z)$$

$$\beta_{0,k}^{\{0,1,0\}} = \frac{3(-1)^k}{2\Delta}\frac{(-(3\Delta+2)\Delta^2 f + 4(\Delta+1)h)c + 3(\Delta^2 f - 4h)}{(2\Delta+1)c^2 - 3(4\Delta+3)c + 18}$$

$$\beta_{0,k}^{\{0,+1/2,-1/2\}} = -\frac{(-1)^k}{4\Delta} \times \frac{((\Delta+1)\Delta f + 4h)c^2 - 3((5\Delta+3)\Delta f + 4h)c + 18\Delta f}{(2\Delta+1)c^2 - 3(4\Delta+3)c + 18}$$

$$\beta_{0,k}^{\{0,\pm 1/2,0,\mp 1/2\}} = \frac{3(-1)^k}{4\Delta c} \times \frac{(\pm(2\Delta+1)g - \Delta^2 f + 4h)c^2 - 3(\pm(4\Delta+3)\,g + \Delta^2 f + 4h)c \pm 18g}{(2\Delta+1)c^2 - 3(4\Delta+3)c + 18}$$

$$\beta_{0,k}^{\{0,0,+1/2,-1/2\}} = \frac{9(-1)^k}{2c}\frac{((3\Delta+1)\Delta f - 4h)c - 6\Delta f}{(2\Delta+1)c^2 - 3(4\Delta+3)c + 18}$$

$$\Phi_0^{\pm\pm 1}(Z) = [\beta_0^{\{0,\pm\pm 1,0\}}\tilde{U}^{\pm\pm}_{-1} + \beta_0^{\{0,\pm 1/2,\pm 1/2\}}\tilde{G}^{\pm}_{-1/2}\tilde{G}^{\pm}_{-1/2} + \beta_0^{\{0,\pm 1/2,0,\pm 1/2\}}\tilde{G}^{+}_{-1/2}\tilde{j}^{-}_{-1/2} + \beta_0^{\{0,0,\pm 1/2,\pm 1/2\}}\tilde{j}^{+}_{-1/2}\tilde{j}^{-}_{-1/2}]\Phi(Z)$$

$$\beta_0^{\{0,\pm\pm 1,0\}} = \frac{3}{2\sqrt{2}\Delta}$$

$$\times \frac{((2\Delta^2-1)g \mp (3\Delta+2)\,\Delta^2 f \mp 4(\Delta+1)h)c - 3(g \pm \Delta^2 f \pm 4h)}{(2\Delta+1)c^2 - 3(4\Delta+3)c + 18}$$

$$\beta_0^{\{0,\pm1/2,\pm1/2\}} = \frac{1}{8\Delta((2\Delta+1)c^2 - 3(4\Delta+3)c+18)}$$
$$\times [(\pm(2\Delta+1)g - (\Delta+1)\Delta f + 4h)c^2$$
$$- 3(\mp 2(3\Delta+2)g + (5\Delta+3)\Delta f - 4h)c + 18(g - \Delta f)]$$

$$\beta_0^{\{0,\pm1/2,0,\pm1/2\}} = -\frac{3}{4\Delta}\frac{(g \pm \Delta^2 f \pm 4h)c - 3((2\Delta+1)g \mp \Delta^2 f \pm 4h)}{(2\Delta+1)c^2 - 3(4\Delta+3)c + 18}$$

$$\beta_0^{\{0,0,\pm1/2,\pm1/2\}} = -\frac{9}{4c}\frac{((3\Delta+1)\Delta f \mp 2\Delta g + 4h)c - 6(\Delta f \mp g)}{(2\Delta+1)c^2 - 3(4\Delta+3)c + 18} \tag{5.7b}$$

for $n = 2$, where $\tilde{L}_n$, $\tilde{G}^{\pm}_{a,r}$,$\tilde{U}_n$, $\tilde{U}^{\pm\pm}_n$, $\tilde{j}^{\pm}_{a,r}$ and $\tilde{j}_n$ are defined by (4.10a) for $\alpha = 0$. We remark that the superconformal family of the superfield Φ exists only for Δ integer. The coefficients f, g, and h are determined by asking for the associativity of the product $\Phi\Phi\Phi$. But we can remark that $\Phi(Z_1)\Phi(Z_2) = \Phi(Z_2)\Phi(Z_1)$, which implies that if Δ is even, we have $g = 0$, and if Δ is odd, we have $f = h = 0$.

6. CONCLUSION

There are two types of $N = 4$ super-W algebra. The first type ($\mu \neq \Delta$) is generated by the stress-tensor superfields $J^{\pm}_a = D^{\pm}_a J$ ($a = 1, 2$) and the primary superfields $\Phi^q_{\pm\nu}$ ($\nu = 0, \ldots, \mu$ if μ is integer, $\nu = 1/2, \ldots, \mu$ if μ is half-integer, and $\mu = 0,1/2, \ldots, \Delta - 1$). The second type ($\mu = \Delta$) is generated by the stress-tensor superfields $J_k = D^+ S_k\, D^- J$ ($k = 1, 2, 3$) and the primary superfields $\Phi^q_{\pm\nu}$ ($\nu = 0, \ldots, \Delta$ if Δ is integer, $\nu = 1/2, \ldots, \Delta$ if Δ is half-integer). So the first type of $N = 4$ super-W algebra admits the large $N = 4$ superconformal algebra as subalgebra, while the second $N = 4$ super-W algebra admits as subalgebra the small $N = 4$ superconformal algebra. In other words, we have shown that the deformation parameter α must be equal to $\alpha^{\pm}(\Delta, \mu, q)$ for the first type and to 1/2 for the second. We remark that for the first type we fix Δ and μ and we vary ν. This means that we have Δ algebras for Δ integer and $\Delta - 1/2$ algebras for Δ half-integer. The difference between these algebras is first the number of primary superfields ($2\mu + 1$) that each algebra contains and second the value of $\alpha = \alpha^{\pm}(\Delta, \mu, q)$. Furthermore, the superfields $\Phi^q_{\pm\nu}$ are null for $q \neq 0$ and the $N = 4$ super-W algebra contains in addition to the generators of the large $N = 4$ superconformal algebra the primary fields W, $F^{\pm}_a$, E_k, H, $H^{\pm\pm}$, $\phi^{\pm}_a$, and ϕ of conformal weight $\Delta + 2$, $\Delta + 3/2$, $\Delta + 1$, $\Delta + 1$, $\Delta + 1$, $\Delta + 1/2$, and Δ, respectively. On the other hand, for the second type the primary fields W^q, $F_a^{q\pm1}$, H^q, and $H^{q\pm2}$ are null fields. So the $N = 4$ super-W algebra contains in addition to the generators of the small $N = 4$ superconformal algebra (L_n, $G^{\pm}_{a,r}$, and $U_{k,n}$) the primary fields E_k^q, $\phi_a^{q\pm1}$, and ϕ^q. For the first type ($\mu \neq \Delta$) in the case of $\mu = 0$ we have determined the descendant fields up to level $n = 2$ for the superconformal family of the identity and up to level $n = 1$ for the superconformal family of the superfield Φ. Notice that for Δ half-integer we have simply the superconformal family of the identity. Hence we have an explicit form of the operator product algebra for $\Delta = 1/2$. For the second type ($\mu = \Delta$) the SOPE $\Phi_i\Phi_j$ with $i, j = 1, 2, 3$ is given in the case of $\mu = \Delta = 1$. The conditions of associativity of the operator product algebra, namely the crossing symmetry and the Jacobi identities, for the first type of $N = 4$ super-W algebra in the case of $\mu = 0$, $\Delta = 1/2$ and for the second type in the case of $\mu = \Delta = 1$ are under study.

APPENDIX

Here we give relations useful for determining the results in the text.

A1. Super-Cauchy Integral

$$\oint_{C_{z_2}} dZ_1\, Z_{12}^{-(n+1)} F(Z_1)$$

$$= \frac{1}{n!}\left(\partial_{z_2}^2 - \tfrac{1}{8}[D_2^+, D_2^-][D_2^+, D_2^-]\right) \partial_{z_2}^n F(Z_2)$$

$$\oint_{C_{z_2}} dZ_1\, \theta_{12}^{\pm a} Z_{12}^{-(n+1)} F(Z_1)$$

$$= \frac{1}{n!}\left(D_2^{\pm a}\, \partial_{z_2} \mp \tfrac{1}{2}[D_2^+, D_2^-] D_2^{\pm a}\right) \partial_{z_2}^n F(Z_2)$$

$$\oint_{C_{z_2}} dZ_1\, (\theta_{12}^{\pm})^2 Z_{12}^{-(n+1)} F(Z_1) = -\frac{1}{n!}(D_2^{\pm})^2\, \partial_{z_2}^n F(Z_2)$$

$$\oint_{C_{z_2}} dZ_1\, (\theta_{12}^{+}\theta_{12}^{-}) Z^{-(n+1)} F(Z_1) = -\frac{1}{2(n!)}[D_2^+, D_2^-]\, \partial_{z_2}^n F(Z_2)$$

$$\oint_{C_{z_2}} dZ_1\, (\theta_{12}^{+} S_k \theta_{12}^{-}) Z_{12}^{-(n+1)} F(Z_1) = \frac{1}{n!}(D_2^+ S_k D_2^-)\, \partial_{z_2}^n F(Z_2)$$

$$\oint_{C_{z_2}} dZ_1\, (\theta_{12}^{+}\theta_{12}^{-})\theta_{12}^{\pm a} Z_{12}^{-(n+1)} F(Z_1) = \mp\frac{1}{n!} D_2^{\pm a}\, \partial_{z_2}^{n} F(Z_2)$$

$$\oint_{C_{z_2}} dZ_1\, \frac{1}{4} (\theta_{12}^{+})^2(\theta_{12}^{-})^2 Z_{12}^{-(n+1)} F(Z_1) = \frac{1}{n!}\, \partial_{z_2}^{n} F(Z_2) \tag{A.1}$$

We can use these integrals to formulate the descendant fields in term of $J_a^{\pm}$ ($a = 1, 2$) for the first type of $N = 4$ super-W algebra ($\mu \neq \Delta$) and in terms of J_k ($k = 1, 2, 3$) for the second type ($\mu = \Delta$).

A2. Recursion Relations

The requirement of associativity with respect to the triple product $J\Phi|\Phi\rangle$ is equivalent to the following relations:

$$\begin{aligned}
&\tilde{L}_n(Z_2)\Phi(Z_1)|\Phi(Z_2)\rangle \\
&\quad = \Big\{ Z_{12}^{n+1}\, \partial_{z_1} + (n+1) Z_{12}^{n} \Big[\Delta + \frac{1}{2}\theta_{12}^{+} D_1^{-} - \frac{1}{2}\theta_{12}^{-} D_1^{+} \Big] \\
&\qquad + n(n+1) Z_{12}^{n-1} \Big[\frac{\alpha}{2} (\theta_{12}^{+})^2 (\theta_{12}^{-})^2 \partial_{z_1} + \alpha(\theta_{12}^{+}\theta_{12}^{-})(\theta_{12}^{+} D_1^{-} + \theta_{12}^{-} D_1^{+}) \Big] \\
&\qquad + \frac{\alpha\Delta}{2}\, n\, (n^2 - 1) Z_{12}^{n-2} (\theta_{12}^{+})^2 (\theta_{12}^{-})^2 \Big\} \Phi(Z_1)|\Phi(Z_2)\rangle \\
&\qquad + \frac{1}{2}(1 + 2\alpha) n(n+1) Z_{12}^{n-1} (\theta_{12}^{+} S_k \theta_{12}^{-}) \Phi_k(Z_1)|\Phi(Z_2)\rangle
\end{aligned}$$

$$\begin{aligned}
&\tilde{G}_{a,r}^{\pm}(Z_2)\Phi(Z_1)|\Phi(Z_2)\rangle \\
&\quad = \Big\{ Z_{12}^{r+1/2} [2\theta_{a12}^{\pm}\, \partial_{z_1} - D_{a1}^{\pm}] + (r + \tfrac{1}{2}) Z_{12}^{r-1/2} \Big[2\Delta\theta_{a12}^{\pm} \\
&\qquad \pm \frac{1}{2}(1 + 2\alpha)(\theta_{12}^{+}\theta_{12}^{-}) D_{a1}^{\pm} \mp \frac{1}{2}(1 + 2\alpha)(\theta_{12}^{\pm})^2 D_{a1}^{\mp} \\
&\qquad + \frac{1}{2}(1 - 2\alpha)(S_k)_{ab} (\theta_{12}^{+} S_k \theta_{12}^{-}) D_1^{\pm b} \mp 4\alpha(\theta_{12}^{+}\theta_{12}^{-})\theta_{a12}^{\pm}\, \partial_{z_1} \Big] \\
&\qquad + \Big(r^2 - \frac{1}{4}\Big) Z_{12}^{r-1/2} \Big[\mp 4\alpha\Delta(\theta_{12}^{+}\theta_{12}^{-})\theta_{a12}^{\pm}
\end{aligned}$$

$$+ \frac{\alpha}{2} (\theta_{12}^{+})^2(\theta_{12}^{-})^2 D_{a1}^{\pm} \Bigg]\Bigg\}\Phi(Z_1)\big|\Phi(Z_2)\rangle$$

$$- (1 + 2\alpha)(-1)^k (S_k)_{ab}\Bigg[\left(r + \frac{1}{2}\right) Z_{12}^{r-1/2}\theta_{12}^{\pm b}$$

$$\mp \left(r^2 - \frac{1}{4}\right) Z_{12}^{r-3/2}(\theta_{12}^{+}\theta_{12}^{-})\theta_{12}^{\pm b}\Bigg]\Phi_k(Z_1)\big|\Phi(Z_2)\rangle$$

$$\tilde{U}_{k,n}(Z_2)\Phi(Z_1)\big|\Phi(Z_2)\rangle$$

$$= \Bigg\{ Z_{12}^{n}(2(\theta_{12}^{+}S_k\theta_{12}^{-})\partial_{z_1} - \theta_{12}^{+}S_k D_1^{-} + \theta_{12}^{-}S_k D_1^{+}) + nZ_{12}^{n-1}[2\Delta(\theta_{12}^{+}S_k\theta_{12}^{-})$$

$$- (\theta_{12}^{+}\theta_{12}^{-})(\theta_{12}^{+}S_k D_1^{-} + \theta_{12}^{-}S_k D_1^{+})]\Bigg\}\Phi(Z_1)\big|\Phi(Z_2)\rangle$$

$$+ \Bigg\{ -2(-1)^k\delta_{ki}Z_{12}^{n} + 2nZ_{12}^{n-1}(-1)^j\varepsilon^{kij}(\theta_{12}^{+}S_j\theta_{12}^{-})$$

$$- \frac{(-1)^k}{2}\,\delta_{ki}(\theta_{12}^{+})^2(\theta_{12}^{-})^2\Bigg\}\Phi_i(Z_1)\big|\Phi(Z_2)\rangle$$

$$\tilde{U}_{n}(Z_2)\Phi(Z_1)\big|\Phi(Z_2)\rangle$$

$$= \Bigg\{ Z_{12}^{n}(2(\theta_{12}^{+}\theta_{12}^{-})\partial_{z_1} - \theta_{12}^{+}D_1^{-} - \theta_{12}^{-}D_1^{+})$$

$$+ nZ_{12}^{n-1}[2\Delta(\theta_{12}^{+}\theta_{12}^{-}) + (\theta_{12}^{+}\theta_{12}^{-})(\theta_{12}^{+}D_1^{-} - \theta_{12}^{-}D_1^{+})]\Bigg\}\Phi(Z_1)\big|\Phi(Z_2)\rangle \quad \text{(A.2a)}$$

$$\tilde{U}_{n}^{\pm\pm}(Z_2)\Phi(Z_1)\big|\Phi(Z_2)\rangle$$

$$= \pm\sqrt{2}\{Z_{12}^{n}[(\theta_{12}^{\pm})^2\partial_{z_1} - \theta_{12}^{\pm}D_1^{\pm}] + nZ_{12}^{n-1}[\Delta(\theta_{12}^{\pm})^2$$

$$\pm (\theta_{12}^{+}\theta_{12}^{-})\theta_{12}^{\pm}D_1^{\pm}]\}\Phi(Z_1)\big|\Phi(Z_2)\rangle$$

$$\tilde{j}_{a,r}^{\pm}(Z_2)\Phi(Z_1)\big|\Phi(Z_2)\rangle$$

$$= \Bigg\{ Z_{12}^{r-1/2}\Bigg[-\frac{1}{2}(\theta_{12}^{+}\theta_{12}^{-})D_{a1}^{\pm} + \frac{1}{2}(\theta_{12}^{\pm})^2 D_{a1}^{\mp}$$

$$\pm \frac{1}{2} (S_k)_{ab}(\theta_{12}^{+} S_k \theta_{12}^{-}) D_1^{\pm b} \Big]$$

$$+ (r - \tfrac{1}{2}) Z_{12}^{r-3/2} \Big[2\Delta(\theta_{12}^{+}\theta_{12}^{-})\theta_{a12}^{\pm}$$

$$\mp \frac{1}{4} (\theta_{12}^{+})^2 (\theta_{12}^{-})^2 D_{a1}^{\pm} \Big]\Big\} \Phi(Z_1) \big| \Phi(Z_2) \rangle$$

$$+ (-1)^k (S_k)_{ab} [Z_{12}^{r-1/2} \theta_{12}^{\pm b} - (r - \tfrac{1}{2}) Z_{12}^{r-3/2} (\theta_{12}^{+}\theta_{12}^{-}) \theta_{12}^{\pm b}] \Phi_k(Z_1) \big| \Phi(Z_2) \rangle$$

$$\tilde{j}_n(Z_2)\Phi(Z_1) \big| \Phi(Z_2) \rangle$$

$$= \frac{1}{2} \{ Z_{12}^{n-1} [(\theta_{12}^{+})^2 (\theta_{12}^{-})^2 \partial_{z_1} + 2(\theta_{12}^{+}\theta_{12}^{-})(\theta_{12}^{+} D_1^{-} + \theta_{12}^{-} D_1^{+})]$$

$$+ \Delta(n - 1) Z_{12}^{n-2} (\theta_{12}^{+})^2 (\theta_{12}^{-})^2 \} \Phi(Z_1) \big| \Phi(Z_2) \rangle$$

$$+ Z_{12}^{n-1} (\theta_{12}^{+} S_k \theta_{12}^{-}) \Phi_k(Z_1) \big| \Phi(Z_2) \rangle$$

with $Z_2 = (0, \theta_2^{+}, \theta_2^{-})$, $n \geq 1$, and $r \geq 1/2$.

The superfields Φ_k ($k = 1, 2, 3$) are given by (4.2). Notice that for the second type ($\mu = \Delta$, $\alpha = 1/2$) of $N = 4$ super-W algebra we have only the three first relations (A.2a).

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